PS Funktionalanalysis II

Gerald Teschl

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1 Übung 1

Problem 1.1 *Show* $(\alpha A)^* = \alpha^* A^*$ *and* $(A+B)^* \supseteq A^* + B^*$ *with equality if one operator is bounded. Give an example where equality does not hold.*

Problem 1.2 *Show*

$$\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp. \tag{1}$$

Problem 1.3 *Show that the formula of partial integration*

$$\int_a^b f(x)g'(x) = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx. \tag{2}$$

holds for $f, g \in AC[a,b]$. *(Hint: Fubini)*

Problem 1.4 *Show that* $H^1(a,b)$ *together with the norm*

$$\|f\|_1 = \int_a^b |f(t)|^2 dt + \int_a^b |f'(t)|^2 dt \tag{3}$$

is a Hilbert space. What is the closure of $C_0^\infty(a,b)$ *in* $H^1(a,b)$*?*

Problem 1.5 *Show that* $f \in H^1(\mathbb{R})$ *satisfies* $\lim_{x\to\pm\infty} f(x) = 0$. *(Hint: Integrate* $\frac{d}{dx}|f(x)|^2 = 2\mathrm{Re}(f(x)^* f'(x))$*.)*

Problem 1.6 *Let* $A = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A) = \{f \in H^2(0,\pi) \,|\, f(0) = f(\pi) = 0\}$ *and let* $\psi(x) = \frac{1}{2\sqrt{\pi}} x(\pi - x)$. *Find the error in the following argument: Since* A *is symmetric we have* $1 = \langle A\psi, A\psi\rangle = \langle \psi, A^2\psi\rangle = 0$.

Problem 1.7 *Show that the kernel of a closed operator is closed.*

Problem 1.8 *Suppose* A *is a closed operator. Show that* A^*A *(with* $\mathfrak{D}(A^*A) = \{\psi \in \mathfrak{D}(A) | A\psi \in \mathfrak{D}(A^*)\}$ *is self-adjoint. (Hint:* $A^*A \geq 0$*.)*

2 Übung 2

Problem 2.1 *What is the spectrum of an orthogonal projection?*

Problem 2.2 *Compute the resolvent of* $Af = f'$, $\mathfrak{D}(A) = \{f \in H^1[0,1] \,|\, f(0) = 0\}$ *and show that unbounded operators can have empty spectrum.*

Problem 2.3 *Compute the eigenvalues and eigenvectors of* $A = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A) = \{f \in H^2(0,\pi) | f(0) = f(\pi) = 0\}$. *Compute the resolvent of* A.

Problem 2.4 *Find a Weyl sequence for the self-adjoint operator* $A = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A) = H^2(\mathbb{R})$ *for* $z \in (0,\infty)$. *What is* $\sigma(A)$*? (Hint: Cut off the solutions of* $-u''(x) = z\,u(x)$ *outside a finite ball.)*

Problem 2.5 *Suppose* A *is bounded. Show that the spectrum of* AA^* *and* A^*A *coincide away from* 0 *by showing*

$$R_{AA^*}(z) = \frac{1}{z}\left(AR_{A^*A}(z)A^* - 1\right), \quad R_{A^*A}(z) = \frac{1}{z}\left(A^*R_{AA^*}(z)A - 1\right). \tag{4}$$

Problem 2.6 *Compute the defect indices of* $A_0 = \mathrm{i}\frac{d}{dx}$, $\mathfrak{D}(A_0) = C_c^\infty((0,\infty))$. *Can you give a self-adjoint extension of* A_0.

3 Übung 3

Problem 3.1 *Show that a self-adjoint operator P is a projection if and only if $\sigma(P) \subseteq \{0, 1\}$.*

Problem 3.2 *Let $\mathfrak{H} = L^2(\mathbb{R})$ and let f be a real-valued measurable function. Show that*

$$P(\Omega) = \chi_{f^{-1}(\Omega)} \tag{5}$$

is a projection valued measure. What is the corresponding operator?

Problem 3.3 *Let $P(\Omega)$ be a projection-valued measure and*

$$P(\lambda) = P((-\infty, \lambda]) \tag{6}$$

the corresponding resolution of the identity. Show that $P(\lambda)$ satisfies:

1. *$P(\lambda)$ is an orthogonal projection.*
2. *$P(\lambda_1) \leq P(\lambda_2)$ (that is $\langle \psi, P(\lambda_1)\psi \rangle \leq \langle \psi, P(\lambda_2)\psi \rangle$) or equivalently $\mathrm{Ran}(P(\lambda_1)) \subseteq \mathrm{Ran}(P(\lambda_2))$ for $\lambda_1 \leq \lambda_2$.*
3. *$\text{s-lim}_{\lambda_n \downarrow \lambda} P(\lambda_n) = P(\lambda)$ (strong right continuity).*
4. *$\text{s-lim}_{\lambda \to -\infty} P(\lambda) = 0$ and $\text{s-lim}_{\lambda \to +\infty} P(\lambda) = \mathbb{I}$*

Problem 3.4 *Let A_1, A_2 be self-adjoint operators with simple spectrum and corresponding maximal spectral measures μ_1 and μ_2. Show that they are unitarily equivalent if μ_1 and μ_2 are mutually absolutely continuous.*

4 Übung 4

Problem 4.1 *Construct a multiplication operator on* $L^2(\mathbb{R})$ *which has dense pure point spectrum.*

Problem 4.2 *Let* $d\mu(\lambda) = \chi_{[0,1]}(\lambda)d\lambda$ *and* $f(\lambda) = \chi_{(-\infty,t]}$, $t \in \mathbb{R}$. *Compute* $f_\star\mu$.

Problem 4.3 *Compute* $\sigma(A)$, $\sigma_{ac}(A)$, $\sigma_{sc}(A)$, *and* $\sigma_{pp}(A)$ *for the multiplication operator* $A = \frac{1}{1+x^2}$ *in* $L^2(\mathbb{R})$. *What is its spectral multiplicity?*

Problem 4.4 *Let* $\mathfrak{H} = L^2(0, 2\pi)$ *and consider the one parameter unitary group given by* $U(t)f(x) = f(x - t \mod 2\pi)$. *What is the generator of* U*?*

5 Übung 5

Problem 5.1 *Compute the resolvent of* $A + \alpha\langle\psi, .\rangle\psi$. *(Hint: Show*

$$(\mathbb{I} + \alpha\langle\varphi, .\rangle\psi)^{-1} = \mathbb{I} - \frac{\alpha}{1 + \alpha\langle\varphi, \psi\rangle}\langle\varphi, .\rangle\psi \tag{7}$$

and use the second resolvent formula.)

Problem 5.2 *Show that* $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $f(n) \mapsto \sum_{j\in\mathbb{N}} k(n+j)f(j)$ *is Hilbert–Schmidt if* $|k(n)| \le C(n)$, *where* $C(n)$ *is decreasing and summable.*

Problem 5.3 *Show that* $A = -\frac{d^2}{dx^2} + q(x)$, $\mathfrak{D}(A) = H^2(\mathbb{R})$ *is self-adjoint if* $q \in L^\infty(\mathbb{R})$. *Show that if* $-u''(x) + q(x)u(x) = zu(x)$ *has a solution for which* u *and* u' *are bounded near* $+\infty$ *(or* $-\infty$*) but* u *is not square integrable near* $+\infty$ *(or* $-\infty$*), then* $z \in \sigma_{ess}(A)$. *(Hint: Use* u *to construct a Weyl sequence by restricting it to a compact set. Now modify your construction to get a singular Weyl sequence by observing that functions with disjoint support are orthogonal.)*

6 Übung 6

Problem 6.1 *Given $\alpha, \beta, \gamma, \delta$, show that there is a function $f \in \mathfrak{D}(\tau)$ restricted to $[c, d] \subseteq (a, b)$ such that $f(c) = \alpha$, $(pf)(c) = \beta$ and $f(d) = \gamma$, $(pf)(c) = \delta$. (Hint: Use the solution formula for the inhomogenous equation in Lemma 9.2.)*

Problem 6.2 *Let $A_0 = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A_0) = \{f \in H^2[0,1] | f(0) = f(1) = 0\}$. and $B = q$, $\mathfrak{D}(B) = \{f \in L^2(0,1) | qf \in L^2(0,1)\}$. Find a $q \in L^1(0,1)$ such that $\mathfrak{D}(A_0) \cap \mathfrak{D}(B) = \{0\}$. (Hint: Start by constructing a function $f \in L^p(0,1)$ which has a pole at every rational number in $[0,1]$.)*

Problem 6.3 *Compute the spectrum and the resolvent of $\tau = -\frac{d^2}{dx^2}$, $I = (0, \infty)$ defined on $\mathfrak{D}(A) = \{f \in \mathfrak{D}(\tau) | f(0) = 0\}$.*