PS Gewöhnliche Differentialgleichungen 2

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1. Can
$$\phi(t) = \begin{pmatrix} \cos(2t) \\ \sin(t) \end{pmatrix}$$
be the solution of an autonomous system $\dot{x} = f(x)$? (Hint: Plot the orbit.) Can it be the solution of $\dot{x} = f(t, x)$?

2. Compute the flow for $f(x) = x^2$ defined on $M = \mathbb{R}$.

3. Show that $\Phi(t, x) = \mathrm{e}^t(1 + x) - 1$ is a flow (i.e., it satisfies $\Phi(0, x) = x$ and $\Phi(s + t, x) = \Phi(t, \Phi(s, x))$). Can you find an autonomous system corresponding to this flow?

4. Find a transformation which straightens out the flow $\dot{x} = x$ defined on $M = \mathbb{R}$.

5. Let $\phi(t)$ be the solution of a first-order autonomous system. Suppose $\lim_{t\to\infty} \phi(t) = x \in M$. Show that x is a fixed point and $\lim_{t\to\infty} \dot{\phi}(t) = 0$. (Hint: Show $\Phi(s, x) = x$.)

6. Let Φ be the flow of some first-order autonomous system.
   (a) Show that if T satisfies $\Phi(T, x) = x$, the same is true for any integer multiple of T. Moreover, show that we must have $T = nT(x)$ for some $n \in \mathbb{Z}$ if $T(x) \neq 0$.
   (b) Show that a point x is fixed if and only if $T(x) = 0$.
   (c) Show that x is periodic if and only if $\gamma_+(x) \cap \gamma_-(x) \neq \emptyset$ in which case $\gamma_+(x) = \gamma_-(x)$ and $\Phi(t + T(x), x) = \Phi(t, x)$ for all $t \in \mathbb{R}$. In particular, the period is the same for all points in the same orbit.

7. A point $x \in M$ is called nonwandering if for every neighborhood U of x there is a sequence of positive times $t_n \to \infty$ such that $\Phi_{t_n}(U) \cap U \neq \emptyset$ for all t_n. The set of nonwandering points is denoted by $\Omega(f)$.
   (a) $\Omega(f)$ is a closed invariant set (Hint: show that it is the complement of an open set).
   (b) $\Omega(f)$ contains all periodic orbits (including all fixed points).
   (c) $\omega_+(x) \subseteq \Omega(f)$ for all $x \in M$.

   Find the set of nonwandering points $\Omega(f)$ for the system $f(x, y) = (y, -x)$.

8. Which of the following equations determine a submanifold of codimension one of $\mathbb{R}^2$?
   (a) $x = 0$.

(b) $x^2 + y^2 = 1$.

(c) $x^2 - y^2 = 1$.

(d) $x^2 + y^2 = 0$.

Which of them is transversal to $f(x, y) = (x, -y)$, $f(x, y) = (1, 0)$, or $f(x, y) = (0, 1)$, respectively.

9. The vector field $f(x, y) = (-y, x)$ has the periodic solution $(\cos(t), \sin(t))$. Compute the Poincaré map corresponding to $\Sigma = \{(x, y) \in \mathbb{R}^2 | x > 0,\ y = 0\}$

10. Consider the system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}, \\ \dot{y} &= x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}. \end{aligned} \tag{1}$$

Show that $x_0(1, 0)$ is not stable even though it satisfies

$$\lim_{t \to \infty} |\phi(t, x) - x_0| = 0 \quad \text{for all } |x - x_0| < 1.$$

(Hint: Show that in polar coordinates the system is given by $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 2\sin(\theta/2)^2$.)

11. Investigate the stability of the fixed points of

$$\dot{x} = \mu x - x^3$$

as a function of μ. Draw phase plots as a function of μ.

12. Show that $L(x, y) = x^2 + y^2$ is a Liapunov function for the system

$$\dot{x} = y, \qquad \dot{y} = -\eta y - x,$$

where $\eta \geq 0$ and investigate the stability of $(x_0, y_0) = (0, 0)$.

13. Consider the mathematical pendulum. If $E = 2$ what is the time it takes for the pendulum to get from $x = 0$ to $x = \pi$?

14. The mathematical pendulum with friction is described by

$$\ddot{x} = -\eta\dot{x} - \sin(x).$$

Is the energy still conserved in this case? Is the fixed point $(\dot{x}, x) = (0, 0)$ (asymptotically) stable? How does the phase portrait change?

Discuss also the linearization

$$\ddot{x} = -\eta\dot{x} - x.$$

15. Suppose all eigenvalues of A satisfy $\mathrm{Re}(\alpha_j) < 0$. Show that every solution of

$$\dot{x}(t) = Ax(t) + g(t)$$

satisfies

$$\lim_{t \to \infty} x(t) = 0.$$

if $\lim_{t \to \infty} |g(t)| = 0$ (Hint: Use the corresponding integral equation.)

16. Find the linearization of
$$f(x) = (x_2, -\sin(x_1)).$$
and determine the stability of $x = 0$ if possible.

17. Consider the system
$$f(x) = (-x_1, x_2 + x_1^2).$$
Find the flow (Hint: Start with the equation for x_1.). Next, find the stable and unstable manifolds. Plot the phase portrait and compare it to the linearization.

18. Consider
$$\dot{x} = -x, \qquad \dot{y} = y^2.$$
Find all invariant smooth manifolds of the form $\{(h(a), a) | a \in \mathbb{R}\}$ which are tangent to E^0.

19. Find $E^{\pm,\alpha}$ for
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
Compute $P^{\pm}$.

20. Investigate the Duffing equation
$$\ddot{x} = -\delta\dot{x} + x - x^3, \quad \delta \geq 0.$$
Determine the stability of the fixed points by linearization. Find the stable and unstable manifolds of the origin in the case $\delta = 0$.

21. Classify the fixed points of the Lorenz equation
$$f(x) = (x_2 - x_1, rx_1 - x_2 - x_1x_3, x_1x_2 - x_3), \quad r > 0,$$
according to stability. At what value of r does the number of fixed points change?

22. Let X be a Banach space and let $A : X \to X$ be a linear operator. Set
$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$
Show that this defines a norm. Moreover, show that
$$\|AB\| \leq \|A\|\|B\|$$
and that $\mathbb{I}+A$ is invertible if $\|A\| < 1$, with inverse given by the **Neumann series**
$$(\mathbb{I} - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$
Furthermore, $\|(\mathbb{I} - A)^{-1}\| \leq (1 - \|A\|)^{-1}$.

23. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism of the form $\varphi(x) = x + h(x)$ with bounded h. Show that $\varphi^{-1}(x) = x + k(x)$, where $k(x)$ is again bounded (with the same bound).

24. Let
$$A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha > 0.$$
Explicitly compute the conjugacy found in the proof of Theorem 7.11.

25. Suppose $\mathrm{div} f = 0$ in some simply connected domain. Show that there is a function $F(x)$ such that $f_1(x) = \frac{\partial F(x)}{\partial x_2}$ and $f_2(x) = -\frac{\partial F(x)}{\partial x_1}$. Show that every orbit $\gamma(x)$ satisfies $F(\gamma(x)) = const$.

26. (Bendixson's criterion) Suppose $\mathrm{div} f$ does not change sign and does not vanish identically in a simply connected region $U \subseteq M$. Show that there are no regular periodic orbits contained (entirely) inside U. (Hint: Suppose there is one and consider the line integral of f along this curve. Recall the Gauss theorem in $\mathbb{R}^2$.)

    Use this to show that
$$\ddot{x} + p(x)\dot{x} + q(x) = 0$$
    has no regular periodic solutions if $p(x) > 0$.

27. If the intersection $\omega_+(x) \cap \omega_-(x) \neq \emptyset$ contains a regular point, then x is periodic.

28. Prove Lemma 8.11.

29. (Volterra principle) Show that for any orbit of the Volterra–Lotka system the time average over one period
$$\frac{1}{T}\int_0^T x(t)dt = 1, \qquad \frac{1}{T}\int_0^T y(t)dt = 1$$
    is independent of the orbit. (Hint: Integrate $\frac{d}{dt}\ln(x(t))$ over one period.)

30. Show that the change of coordinates $x = \exp(q)$, $y = \exp(p)$ transforms the Volterra–Lotka system into a Hamiltonian system with Hamiltonian $H(p, q) = L(\exp(q), \exp(p))$.

    Moreover, use the same change of coordinates to transform
$$\dot{x} = (1 - y - \lambda x)x, \quad \dot{y} = \alpha(x - 1 - \mu y)y, \qquad \alpha, \lambda, \mu > 0.$$
    Then use Bendixson's criterion (Problem 26) to show that there are no periodic orbits.

31. Suppose you have two species x and y such that one inhibits the growth of the other. A simple model describing such a situation would be
$$\begin{array}{l} \dot{x} = (A - By)x \\ \dot{y} = (C - Dx)y \end{array}, \qquad A, B, C, D > 0.$$
    Find out as much as possible about this system.

32. The equation
$$\ddot{x} + g(x)\dot{x} + x = 0$$
is also often called Liénard's equation. Show that it is equivalent to $\dot{x} = y - f(x)$, $\dot{y} = -x$ if we set $y = \dot{x} + f(x)$, where $f(x) = \int_0^x g(t)dt$.

33. Show that
$$\dot{z} = z(\mu - (\alpha + \mathrm{i}\beta)|z|^2), \qquad \mu \in \mathbb{R}, \alpha, \beta > 0,$$
where $z(t) = x(t) + \mathrm{i}y(t)$, exhibits a Hopf bifurcation at $\mu = 0$. (Hint: Use polar coordinates $z = r\mathrm{e}^{\mathrm{i}\varphi}$.)

34. Show that a closed invariant set which has a dense orbit is topologically transitive.

35. Suppose $\gamma_\sigma(x)$ is contained in a compact set. Show that
$$\lim_{t\to\infty} d(\Phi(t,x), \omega_\sigma(x)) = 0.$$

36. Suppose E is a trapping region and let $\Lambda = \omega_+(E)$. Then
$$W^+(\Lambda) = \{x \in M | \omega_+(x) \subseteq \Lambda,\ \omega_+(x) \neq \emptyset\}.$$
(Hint: Previous problem.)

37. Solve the Lorenz equation for the case $\sigma = 0$.

38. Investigate the Lorenz equation for the case $r = \infty$ as follows. First introduce $\varepsilon = r^{-1}$. Then use the change of coordinates $(t, x, y, x) \mapsto (\tau, \xi, \eta, \zeta)$, where $\tau = \varepsilon^{-1}t$, $\xi = \varepsilon x$, $\eta = \sigma\varepsilon^2 y$, and $\zeta = \sigma(\varepsilon^2 z - \varepsilon)$.

    Show that the resulting system for $\varepsilon = 0$ is given by
$$\xi' = \eta, \quad \eta' = -\xi\zeta, \quad \zeta' = \eta\xi,$$
    which has two conserved quantities
$$\xi^2 - 2\zeta = 2\alpha, \quad \eta^2 + \zeta^2 = \beta.$$
    Derive the single third order equation $\xi''' = -(\frac{3}{2}\xi^2 - \alpha)\xi'$. Integrate this equation once and observe that the result is of Newton type. Conclude that ξ (and hence η and ζ) are always periodic.

39. Consider the logistic map L_μ for $\mu = 1$. Show that $W^+(0) = [0, 1]$.

40. Determine the stability of all fixed points of the logistic map L_μ, $0 \le \mu \le 4$.

41. Consider the logistic map L_μ for $\mu = 4$. show that 0 is a repelling fixed point. Find an orbit which is both forward and backward asymptotic to 0.

42. Find an explicit formula for the **Fibonacci numbers** defined via
$$x(m) = x(m-1) + x(m-2), \qquad x(1) = x(2) = 1.$$

43. Show item (i) from the proof of Lemma 11.1.

44. Show item (ii) from the proof of Lemma 11.1.

45. Show that a closed invariant set which has a dense orbit is topologically transitive.

46. Show that T_2 and L_4 are topologically equivalent via the map $\varphi(x) = \sin(\frac{\pi x}{2})^2$ (i.e., show that $\varphi : [0,1] \to [0,1]$ is a homeomorphism and that $\varphi \circ T_2 = L_4 \circ \varphi$).

47. Show that two different ternary expansions define the same number, $\sum_{n\in\mathbb{N}} x_n 3^{-n} = \sum_{n\in\mathbb{N}} y_n 3^{-n}$, if and only if there is some $n_0 \in \mathbb{N}$ such that $x_j = y_j$ for $n < n_0$, $x_j = y_j \pm 1$ for $n = n_0$, and $x_j = y_j \mp 2$ for $n > n_0$. Show that every $x \in [0,1]$ has a unique expansions if the expansions which end in 1 or $1\overline{2}$ are excluded.

48. Show that for $\mu = 3$ we have $\Lambda_n = \{x | x_j \neq 1,\ 1 \leq j \leq n\}$, where x_j are the digits in the ternary expansion as in the previous problem.