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Scattering Theory for One-Dimensional Schrödinger Operators with Measures

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Abstract

The basic quantum scattering formalism for one-dimensional Schrödinger operators is generalized to potentials given as measures. This is done for direct scattering and inverse scattering. We see that most of the classical results for real potentials stay valid in the case of the measure-valued time-independent Schrödinger equation.

Zusammenfassung

Die grundlegenden Resultate der eindimensionalen Streutheorie für Schrödinger Operatoren werden für maßwertige Potentiale verallgemeinert. Dies wird für direkte und inverse Streuung durchgeführt und es stellt sich heraus, dass die meisten klassischen Resultate für reelle Potentiale gültig bleiben.

Contents

0 Introduction

The aim of the present thesis is to generalize direct and inverse scattering theory for the one-dimensional Schrödinger operator to work with measure-valued potentials. The initial point will be the time-independent Schrödinger equation

$$-f'' + f d\mu = k^2 f$$

with μ being a σ-finite Borel measure. This generalization for the involved potential covers the case of point interactions which is an important model in physics. We use the Marchenko approach [6] which deals with a real-valued potential q satisfying some essential short-range conditions.

Chapter 1 starts to recall some facts about measure theory. This is necessary for the formalism we use to define our measure-valued Schrödinger equation. The main concept behind this generalization is based on using the Radon-Nikodym derivative instead of the classical derivative. It can be done for functions that are absolutely continuous with respect to the corresponding measures. This allows us to use the standard notation and results from the theory of ordinary differential equation, but actually meaning an integral equation using a non-continuous measure for describing the potential. For more information about this formalism we refer to [5].

Chapter 2 deals with direct scattering theory for the one-dimensional Schrödinger operator for its time-independent equation. We begin this chapter with some basic assumptions which are relevant for the whole theory. Especially the so-called short-range condition $\int_{-\infty}^{\infty}(1+|x|^p)d|\mu|(x) < \infty$ for $p \geq 0$ is important for the existence and asymptotic completeness of the wave operators. For solving the Schrödinger equation and showing uniqueness of its solution we use the technique of successive approximation and the Weissinger fixpoint theorem. This is technically done for the corresponding Volterra integral equations which also allows us to obtain many estimates for their solutions. We consider the case in which our complex parameter k has $\mathrm{Im}(k) > 0$ and the case in which $\mathrm{Im}(k) \leq 0$ and get the so-called Jost solutions $f_\pm(k,x)$ and $g_\pm(k,x)$. Then we show that the Wronskians $W(g_\pm, f_\pm)$ are independent of x and therefore we can express the coefficients of the solutions in terms of those. Later we see that the Wronskians are suitable to define the reflection and transmission coefficients which allows a better descriptive physical interpretation of the scattering process. We consider the associated Schrödinger operator $H = -\frac{d^2}{dx^2} + d\mu$ and look at its spectrum. The finite many eigenvalues of H are given through the zeros of a function which only depends on the above Wronskians. Turning to the continuous spectrum allows us to define the scattering matrix which is given through

$$S(k) = \begin{pmatrix} T_-(k) & R_+(k) \\ R_-(k) & T_+(k) \end{pmatrix} \text{ for } k \in \mathbb{R}$$

where $T_\pm$ denote the transmission coefficients and $R_\pm$ denote the reflection coefficient. We show that $S(k)$ is a continuous unitary operator. For a physical interpretation we define the wave functions $\psi_\pm$ of H and look at its asymptotic behavior.

The inverse scattering theory is treated in chapter 3. We show that it is possible to reconstruct the scattering matrix with the information from one of the two scattering data. The proof for this is mainly based on the Cauchy's integral formula. In the next step we will construct the Marchenko equation which is fundamental for the inverse scattering process. It is a Fredholm integral equation connecting the scattering data with functions $B_\pm$ which contain all information about the involved potential. We study the conditions under which the Marchenko equation has a unique solution.

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1 A Brief Summary of Basic Facts

1.1 Measure Theory

Here some basic facts about measure theory are recalled. They are essentially collected from [10] (Chapter 7), [5] (Appendix A), [8], [9] and [4].

In the following we will deal with Lebesgue-Stieltjes measures generated by non-decreasing function $m : \mathbb{R} \to \mathbb{R}$ on an interval of the real line. For any interval I its measure can be defined via

$$\mu(I) \quad = \quad \begin{cases} m(b+) - m(a+), \; I = (a,b], \\ m(b+) - m(a-), \; I = [a,b], \\ m(b-) - m(a+), \; I = (a,b), \\ m(b-) - m(a-), \; I = [a,b), \end{cases} \tag{1.1}$$

where $m(a\pm) = \lim_{\epsilon \searrow 0} m(a \pm \epsilon)$. For infinite endpoints we use $m(\pm\infty) = \lim_{x\to\pm\infty} m(x)$ and for singletons we set $\mu(\{a\}) = m(a+) - m(a-)$. This can be extended to a measure on the σ-Algebra of $\mathbb{R}$ by the following theorem:

Theorem 1.1. *For every nondecreasing function $m : \mathbb{R} \to \mathbb{R}$ there exists a unique Borel measure μ which extends (1.1). Two different functions generate the same measure if and only if the difference is a constant away from the discontinuities.*

For details see [10] chapter 7.1.

The substitution rule for integrals can also be generalized to work for arbitrary nondecreasing function.

Lemma 1. *Let m,n be nondecreasing functions on $\mathbb{R}$ and h monotone. We have:*

$$\int_{\mathbb{R}} (h \circ m) d(n \circ m) \leq \int_{hull(Ran(m))} h dn. \tag{1.2}$$

Corollary 1. *One special application of the previous Lemma is the substitution rule with the Borel measure $\mu(x)$ and nondecreasing $a, b \in L^\infty$:*

$$\int a(b(x)) b'(x) d\mu(x) \leq \int a(y) d\mu(y). \tag{1.3}$$

Proof. We use the substitution with the generalized inverse $m = b^{-1}(y)$ in Lemma (1). For the distribution function $n(x) = \mu(x)$ we get the desired result. □

Absolute Continuous Measures and Functions; Theorem of Radon-Nikodym

Let μ, ν be two measures on a measurable space (X, Σ). We call ν absolutely continuous with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$ for every measurable set $A \in \Sigma$.

The theorem of Radon-Nikodym will also be important.

Theorem 1.2. *(Radon-Nikodym)*
Let μ, ν be two σ-finite measures on a measurable space (X, Σ). Then ν is absolutely continuous with respect to μ if and only if there is a positive measurable function f such that

$$\nu(A) = \int_A f d\mu$$

for every $A \in \Sigma$. The function f is determined uniquely almost everywhere with respect to μ and is called the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ of ν with respect to μ.

A function $f : [a, b] \to \mathbb{C}$ is called absolutely continuous if for every $\epsilon > 0$ there is corresponding $\delta > 0$ such that

$$\sum_k |y_k - x_k| < \delta \Rightarrow \sum_k |f(y_k) - f(x_k)| < \epsilon$$

for every countable collection of pairwise disjoint intervals $(x_k, y_k) \subset [a, b]$. They are denoted by $AC[a, b]$ and are a subset of the continuous functions $C[a, b]$. We can give an equivalent definition using the following theorem:

Theorem 1.3. *A function $f : [a, b] \to \mathbb{C}$ is absolutely continuous if and only if it is of the form*

$$f(x) = f(a) + \int_a^x g(y) dy$$

for some integrable function g. In this case f is differentiable a.e. with respect to Lebesgue measure and $f'(x) = g(x)$.

Now we switch to the more general case of a Lebesque-Stiltjes measure μ: We will denote the set of right-continuous functions, which are locally absolute continuous with respect to μ as $AC_{loc}((a, b); \mu)$ where (a, b) is an arbitrary intervall of $\mathbb{R}$. These functions are exactly the functions f, which can be written in the form

$$f(x) = f(c) + \int_c^x h(t) d\mu(t), x \in (a, b)$$

with $h \in L_{loc}((a, b), \mu)$. The integral has to be interpreted as

$$\int_c^x h(t) d\mu(t) = \begin{cases} \int_{(c,x]} h(t) d\mu(t) & \text{if } x > c \\ 0 & \text{if } x = c \\ -\int_{(c,x]} h(t) d\mu(t) & \text{if } x < c. \end{cases}$$

In this case the function h is the Radon-Nikodym derivative of f with respect to the measure μ. It is uniquely defined and is denoted by

$$\frac{df}{d\mu} = h.$$

1.2 Functional Analysis

We will need a generalization of the Banach fix-point theorem which can be found in [11] (Theorem 2.4).

Theorem 1.4. *(Weissinger)*
Let C be a nonempty closed subset of a Banach space X. Suppose $K : C \longrightarrow C$ satisfies

$$\|K^n(x) - K^n(y)\| \leq \Theta_n \|x - y\| \; x, y \in C,$$

with $\sum_{n=1}^{\infty} \Theta_n < \infty$. Then K has a unique fixed point $\tilde{x}$ such that

$$\|K^n(x) - \tilde{x}\| \leq \left(\sum_{j=n}^{\infty} \Theta_j \right) \|K(x) - x\|, \; x \in C.$$

2 Direct Scattering

This and the following chapter are based on [15] which covers the case of a real-valued potential. We will generalize the results for a potential which is allowed to be a measure.

2.1 Some Notation and Basic Assumptions

From now on, we assume that $(\mathbb{R}, \mathcal{B}, \mu)$ is a measure space based on the real numbers $\mathbb{R}$, the Borel sets $\mathcal{B}$ and a signed locally finite Borel measure $\mu(x)$. We define the L^p spaces as:

$$L^p(\mathbb{R}, \mathcal{B}, \mu) := \{f \text{ measurable }, \|f(x)\|_p < \infty\} \tag{2.1}$$

with the p-norm $\|f\|_p := (\int_{-\infty}^{\infty} |f| d\mu(x))^{\frac{1}{p}}$ for $0 < p < \infty$ and for the case $p = \infty$ we have $\|f\|_\infty := ess \sup \|f(x)\|$.

Because μ is a signed measure, we sometimes have to use the total variation and denote it as $|\mu|$.

In some cases we have to restrict the measure μ to suit the typical short-range condition

$$\int_{-\infty}^{\infty} (1 + |x|^p) d|\mu|(x) < \infty \tag{2.2}$$

for a $p \geq 0$.

Now let k be a complex parameter and $f \in L^\infty$. We would like to solve the time-independent Schrödinger equation

$$-f'' + f d\mu = k^2 f \tag{2.3}$$

with the measure-valued coefficient $d\mu$. This informally given expression has to be interpreted as an integral equation (more in [5]). By integrating twice and using Fubini's theorem it can be read as

$$f(x) = c_1 + c_2 x + \int_{(a,x]} (x - t) f(t) d\mu(t) - \int_{(a,x]} (x - t) f(t) k^2 dt$$

with the boundary conditions

$$c_1 = f(x_0) \text{ and } c_2 = f'(x_0) - \int_a^{x_0} f(t)d\mu(t).$$

We start with a proposition about the Volterra integral equation:

2.2 Volterra Integral Equation

Proposition 1. *Suppose $f \in L^\infty(-\infty, x]$, $x \in \mathbb{R}$, and for $x' \leq x$ let $K(x, x')$ satisfy:*

.) $|K(x, x')| \leq K_1(x, x')$ with $K_1(., x')$ is non-decreasing for each $x' \leq x$

.) $K_1(x, .) \in L^1(-\infty, x]$ respective μ $\forall x \in \mathbb{R}$

.) $K_1(., x') \in AC_{loc}$, $K_1(x', x') = 0$.

Then the Volterra integral equation

$$g(x) = f(x) + \int_{-\infty}^{x} K(x, x')g(x')d\mu(x') \tag{2.4}$$

has a unique solution g with the properties:

(i) $g - f \in C^0$, if $K(., x) \in C^0(x', \infty)\forall x' \in \mathbb{R}$

(ii) $|g(x) - f(x)| \leq \sup_{(-\infty,x]} |f(x')| \int_{-\infty}^{x} K_1(x, x')d|\mu|(x') \exp(\int_{-\infty}^{x} K_1(x, x')d|\mu|(x'))$

(iii) $g \in L^\infty(\mathbb{R})$, if $f \in L^\infty(\mathbb{R})$ and $\int_{-\infty}^{x} K_1(x, x')d|\mu|(x') \in L^\infty(\mathbb{R})$.

Proof. We use the method of successive approximation by changing it to a fix point problem and use the Weissinger theorem (1.4) for existence and uniqueness. We define the operators $V : L^\infty(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$ by

$$V(g)(x) = \int_{-\infty}^{x} K(x, x')g(x')d\mu(x')$$

and

$$T(g) = f + V(g) \tag{2.5}$$

with $V^0 = Id$. This is well defined, because for $f \in L^\infty$ one can see that $V(f) \in L^\infty$ again:

$$\begin{aligned}
|V(f)(x)| &\le \sup_{x'\in(-\infty,x]} |f(x')| \int_{-\infty}^{x} K_1(x,x')d|\mu|(x') \\
&\le \sup_{x'\in(-\infty,y]} |f(x')| \int_{-\infty}^{x} K_1(x,x')d|\mu|(x') \\
&\le \sup_{x'\in(-\infty,x]} |f(x')| \int_{-\infty}^{x} K_1(y,x')d|\mu|(x') \\
&\le \sup_{x'\in(-\infty,x]} |f(x')| \int_{-\infty}^{y} K_1(y,x')d|\mu|(x') \\
&=: M(y) < \infty \text{ for } x \le y.
\end{aligned} \tag{2.6}$$

By induction we get

$$T^N g = \sum_{k=0}^{N-1} V^k f + V^N g$$

for $N \in \mathbb{N}$.

Now the solution of the integral equation (2.4) is equivalent to $Tg = g$. We have to show that there is a fix point of T.

For using Theorem (1.4) we need a function Θ_n with $\sum_{k=0}^{\infty} \Theta_k < \infty$ for which

$$\|T^n(g) - T^n(h)\| \le \Theta_n \|g-h\| \quad g,h \in L^\infty(\mathbb{R}). \tag{2.7}$$

Here $\|.\|$ means the essential supreme norm. Let $g,h \in L^\infty$.

$$\begin{aligned}
\|T^n(g) - T^n(h)\| &= \|(\sum_{k=0}^{n-1} V^k f + V^n g) - (\sum_{k=0}^{n-1} V^k f + V^n h)\| = \|V^n(g) - V^n(h)\| \\
&= \| \int_{-\infty}^{x} K(x,x_1) \int_{-\infty}^{x_1} K(x_1,x_2) ... \int_{-\infty}^{x_{n-1}} K(x_{n-1},x_n) g d\mu(x_n) ... d\mu(x_1) \\
&- \int_{-\infty}^{x} K(x,x_1) \int_{-\infty}^{x_1} K(x_1,x_2) ... \int_{-\infty}^{x_{n-1}} K(x_{n-1},x_n) h d\mu(x_n) ... d\mu(x_1) \| \\
&= \int_{-\infty}^{x} K(x,x_1) \int_{-\infty}^{x_1} K(x_1,x_2) ... \int_{-\infty}^{x_{n-1}} K(x_{n-1},x_n)(g-h) d\mu(x_n) ... d\mu(x_1) \\
&\le \int_{-\infty}^{x} |K(x,x_1)| \int_{-\infty}^{x_1} |K(x_1,x_2)| ... \int_{-\infty}^{x_{n-1}} |K(x_{n-1},x_n)| \; |g-h| \; d|\mu|(x_n) ... d|\mu|(x_1) \\
&\le \left(\int_{-\infty}^{x} K_1(x_1,x_2) ... \int_{-\infty}^{x_{n-1}} K_1(x_{n-1},x_n) d|\mu|(x_n) ... d|\mu|(x_1) \right) \; \|g-h\| \\
&\le \frac{1}{n!} \left(\int_{-\infty}^{x} K_1(x,x') d\mu(x') \right)^n \; \|g-h\|
\end{aligned} \tag{2.8}$$

Note that in (2.8) we use the generalized substitution rule from Corollary 1.1 and apply it n times for $a(x) = Id$ and $b(x) = \int_{-\infty}^{x} K(x,x')$. I.e. for the first step:

$$\begin{aligned}
& \int_{-\infty}^{x_{n-2}} K_1(x_{n-2},x_{n-1}) \int_{-\infty}^{x_{n-1}} K_1(x_{n-1},x_n) d|\mu|(x_n)\ d|\mu|(x_{n-1}) \\
\leq\ & \int_{-\infty}^{x_{n-2}} Id(y) d|\mu|(y) \text{ with } y = \int_{-\infty}^{x_{n-1}} K(x_{n-1},x_n') d|\mu|(x_n) \\
\leq\ & \int_{-\infty}^{x_{n-2}} y d|\mu|(y) \\
\leq\ & \frac{y^2}{2} = \frac{(\int_{-\infty}^{x_{n-1}} K(x_{n-1},x_n') d|\mu|(x_n))^2}{2}
\end{aligned}$$

We repeat this step until we have the desired result.

The function $\Theta_n = \frac{1}{n!}\left(\int_{-\infty}^{x} K_1(x,x') d|\mu|(x')\right)^n$ will also converge as a series with

$$\sum_{n=1}^{\infty} \frac{1}{n!}\left(\int_{-\infty}^{x} K_1(x,x') d\mu(x')\right)^n = \exp\left(\int_{-\infty}^{x} K_1(x,x') d|\mu|(x')\right) < \infty. \tag{2.9}$$

Theorem (1.4) can be used to show that there is a unique fix point of T. Hence the Volterra integral equation (2.4) has a unique solution.

For proofing (ii) we use the same estimates and take the sum over $|V^n f|$.

$$\begin{aligned}
|g(x) - f(x)| = \sum_{n=1}^{\infty} |V^n f| \leq \|f\| \sum_{k=1}^{\infty} \left(\int_{-\infty}^{x} K_1(x,x') d|\mu|(x')\right)^n \\
\leq \|f\| \int_{-\infty}^{x} K_1(x,x') d|\mu|(x') \exp\left(\int_{-\infty}^{x} K_1(x,x') d|\mu|(x')\right) < \infty
\end{aligned}$$

Hence the series converges absolutely and uniformly on every compact interval in $\mathbb{R}$. □

2.3 Solving the Schrödinger Equation

Theorem 2.1. *(i) Suppose μ satisfies property (2.2) with $p = 1$. For each k with* $\mathrm{Im}(k) > 0$ *the integral equations*

$$f_\pm(k,x) = e^{\pm ikx} - \int_x^{\pm\infty} \frac{\sin(k(x-x'))}{k} f_\pm(k,x')d\mu(x') \tag{2.10}$$

have unique solutions defined everywhere in $\mathbb{R}$*, which solve the Schrödinger equation (2.3).*

For each x the functions $f_\pm(k,x)$,$f'_\pm(k,x)$ are analytic in the upper half plane $\mathrm{Im}(k) > 0$ *and continuous in* $\mathrm{Im}(k) \geq 0$*. We have the following estimates:*

$$|f_\pm(k,x) - e^{\pm ikx}| \leq \frac{const}{|k|}\exp(\frac{const}{|k|})e^{\mp(\mathrm{Im}(k))x}, k \neq 0 \tag{2.11}$$

$$|f_\pm(k,x)| \leq const(1+\max\{x,0\})e^{\mp\mathrm{Im}(k)x} \tag{2.12}$$

$$|f'_\pm(k,x)| \leq const(\frac{1+|k|}{|k|})e^{\mp\mathrm{Im}(k)x}, k \neq 0 \tag{2.13}$$

$$|f'_\pm(k,x)| \leq const(1+|k|+|k||x|)e^{\mp\mathrm{Im}(k)x} \tag{2.14}$$

(ii) If μ satisfies property (2.2) with $p = 2$, then $\frac{\partial}{\partial k}f_\pm(k,x) = \dot{f}_\pm(k,x)$ exists for $\mathrm{Im}(k) \leq 0$ *and is continuous as a function of k. We get the following estimates:*

$$|\frac{\partial}{\partial k}(e^{\mp ikx}f_\pm(k,x))| \leq const(1+x^2) \tag{2.15}$$

$$|\frac{\partial}{\partial k}f_\pm(k,x)| \leq const(1+x^2)e^{\mp\mathrm{Im}(k)x} \tag{2.16}$$

$$|\frac{\partial^2}{\partial x\partial k}(e^{\mp ikx}f_\pm(k,x))| \leq const(1+|x|) \tag{2.17}$$

$$|\frac{\partial^2}{\partial x\partial k}f_\pm(k,x)| \leq const(1+|k|)(1+x^2)e^{\mp\mathrm{Im}(k)x}. \tag{2.18}$$

Proof. We will do the proof only for f_-.
(ii)
We define $m(k,x) = e^{ikx}f_-(k,x)$ and then the integral equation (2.10) changes to

$$m(k,x) = 1 + \int_{-\infty}^x (\frac{e^{2ik(x-x')}-1}{2ik})m(k,x')d\mu(x'). \tag{2.19}$$

At the next step we would like to apply Proposition(1). We need to estimate $K(x,x') = \frac{e^{2ik(x-x')}-1}{2ik}$. We have

$$|K(x,x') \leq \frac{1}{|k|}, \text{ i.e., } \int_{-\infty}^x |K(x,x')|d|\mu|(x') \leq \frac{\|1\|_1}{|k|}, k \neq 0 \tag{2.20}$$

or

$$|K(x,x')| \leq (x-x'), \text{ i.e., } \int_{-\infty}^x |K(x,x')|d|\mu|(x) \leq \max\{x,0\}\|1\|_1 + \|x\|_1 \tag{2.21}$$

for $x - x' \geq 0$ and $\mathrm{Im}(k) \geq 0$. Both estimates on $|K(x,x')|$ satisfy the condition on $K_1(x,x')$ in Proposition (1). Using the estimate (2.20) shows that for $k \neq 0$ m and f_- are uniquely defined functions. Estimate (2.11) holds and we show estimate (2.13) by inserting it into

$$m'(k,x) = \int_{-\infty}^{x} e^{2ik(x-x')} m(k,x') d\mu(x'). \tag{2.22}$$

With the help of estimate (2.21) we can show that

$$|m(k,x) - 1| \leq \gamma(x) e^{\gamma(x)} \tag{2.23}$$

with

$$\gamma(x) = \int_{-\infty}^{x} (x-t) d|\mu|(t)$$

for all k and prove existence and uniqueness also for $k = 0$. Hence m, m' and therefore $f_\pm, f'_\pm$ are absolutely continuous as functions of x. By direct calculation one can show that $f_\pm$ satisfies the Schrödinger equation. Because of the locally uniform convergence of the series for m, the analyticity of m in $\mathrm{Im}(k) > 0$ and continuity in $\mathrm{Im}(k) \leq 0$ are proven. Also for $f_\pm$. At this point only estimate (2.12) and (2.14) are left: For (2.12) note that $|m(k,x)| \leq 1 + \gamma(0) e^{\gamma(0)}$ for $x \leq 0$ by (2.23). Let $x > 0$. We obtain using estimate (2.21) for some $K \geq 0$

$$\begin{aligned} |m(k,x)| &\leq 1 + \int_{-\infty}^{x} (x - x')|m(k,x')| d|\mu|(x') \\ &\leq 1 + \int_{-\infty}^{x} |x'||m(k,x')|\, d|\mu|(x') + \int_{-\infty}^{x} x|m(k,x')| d|\mu|(x') \\ &\leq K + \int_{-\infty}^{x} x(1+|x'|) \frac{|m(k,x')|}{(1+|x'|)} d|\mu|(x'). \end{aligned}$$

We define $M(k,x) = \frac{m(k,x)}{K(1+|x|}$) and get $M(k,x) \leq \int_{-\infty}^{x} (1+|x'|)|M(k,x')|d|\mu|(x')$. Inserting this inequality repeatedly into itself one obtains

$$M(k,x) \leq \sum_{j=0}^{n-1} M_j(k,x) + \int_{-\infty}^{x} (1+|x_1|) \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_{n-1}} (1+|x_n|) M(k,x_n) d|\mu|(x_n) \dots d|\mu|(x_1) \tag{2.24}$$

where $M_0(k,x) = 1$
and
$M_n(k,x) = \int_{-\infty}^{x} (1+|x'|) M_{n-1}(k,x') d|\mu|(x') \leq \frac{1}{n!} (\int_{-\infty}^{x} (1+|x'|) d|\mu|(x'))^n$.

The last term can be estimated by

$$\frac{1+\gamma(x)e^{\gamma}(x)}{K(1+x)} \frac{1}{n!} \left(\int_{-\infty}^{x} (1+|x'|) d|\mu|(x') \right)^n \tag{2.25}$$

which tends to zero as n tends to infinity locally uniformly in x. Therefore on any bounded interval independently of x

$$M(k,x) \leq \sum_{n=0}^{\infty} \leq \exp\left(\int_{-\infty}^{x} (1+|x'|)d|\mu|(x')\right) \leq const. \tag{2.26}$$

This implies $m(k,x) \leq const(1+x)$ for $x>0$ and proves estimate (2.12). Using this estimate in (2.22) shows estimate (2.14).

(ii)

We differentiate (2.19) with respect to k and get

$$\dot{m}(k,x) = \int_{-\infty}^{x} \frac{\partial}{\partial k}(\frac{e^{2ik(x-x')}-1}{2ik})m(k,x')d\mu(x') + \int_{-\infty}^{x} \frac{e^{2ik(x-x')}-1}{2ik}\dot{m}(k,x')d\mu(x'). \tag{2.27}$$

Note that the absolute of

$$\frac{\partial}{\partial k}(\frac{e^{2ik(x-x')}-1}{2ik}) = \frac{\partial}{\partial k}\int_0^{x-x'} e^{2ikt}dt = \int_0^{x-x'} 2ite^{2ikt}dt$$

can be estimated by $(x-x')^2$. We use this to estimate the first term in (2.27): Suppose $x>0$. Then

$$\begin{aligned}\int_{-\infty}^{x} x'^2|m(k,x)|d\mu(x') \leq const(\int_{-\infty}^{0} x'^2 d|\mu|(x') + x^2\int_0^x (1+x')d|\mu|(x'))\\ \leq const(1+x^2).\end{aligned}$$

Hence

$$\int_{-\infty}^{x}(x-x')^2|m(k,x')|d\mu(x') \leq \int_{-\infty}^{x}(2x^2x'^2)|m(k,x')|d|\mu|(x') \leq const(1+x^2).$$

For $x<0$ we have

$$\int_{-\infty}^{x}(x-x'^2)|m(k,x')|\mu(x) \leq \int_{-\infty}^{x} x'^2|m(k,x')|\ d|\mu|(x') \leq const.$$

The first term in (2.27) is in $L^\infty(-\infty,x]$ and (2.27) is read as an integral equation to which Proposition(1) is applied. With this and using (2.23) we get the estimate

$$|\dot{m}(k,x)| \leq const(1+x\max\{x,0\})(1+\gamma(x)e^{\gamma}(x)) \tag{2.28}$$

which holds independently of x and proves that $\dot{m}$ is continuous everywhere in $\mathrm{Im}(k) \geq 0$ including $k=0$.

To obtain estimate (2.15) consider (2.27) to get

$$\begin{aligned}|\dot{m}(k,x)| &\leq const(1+x^2) + \int_{-\infty}^{x}(x-x')|\dot{m}(k,x')|d|\mu|(x')\\ &\leq const(1+x^2) + |x|\int_{-\infty}^{x}|\dot{m}(k,x')|d|\mu|(x') + \int_{-\infty}^{0}|x'||\dot{m}(k,x)|d|\mu|(x').\end{aligned}$$

We define $h(k,x) = \frac{|\dot{m}(k,x)|}{K(1+x^2)}$ for an appropriate K and use (2.28) for

$$h(k,x) \leq 1 + \int_{-\infty}^{x} (1 + x'^2) d|\mu|(x').$$

We repeat the analysis which leads to (2.12) and obtain

$$h(k,x) \leq \exp\left(\int_{-\infty}^{x} (1 + x'^2) d|\mu|(x')\right) \leq const$$

and hence the estimates (2.15) and (2.16). For obtaining the estimates (2.17) and (2.18) we differentiate (2.27) with respect to k:

$$|\dot{m}'(k,x)| \leq \int_{-\infty}^{x} 2(x-x')|m(k,x')|d|\mu|(x') + \int_{-\infty}^{x} \dot{m}(k,x')d|\mu|(x')$$
$$\leq const(1 + \max\{x, 0\}). \tag{2.29}$$

□

Like above we consider functions $g_{\pm}(k,x)$ defined by the integral equation for $\mathrm{Im}(k) \leq 0$:

$$g_{\pm}(k,x) = e^{\mp ikx} - \int_{x}^{\pm\infty} \frac{\sin(k(x-x'))}{k} g_{\pm}(k,x')d\mu(x'). \tag{2.30}$$

There are some relations between $f\pm$ and $g_{\pm}$. They can be expressed by

$$g_{\pm}(k,x) = f_{\pm}(k^*,x)^* \text{ for } \mathrm{Im}(k) \leq 0 \tag{2.31}$$

$$g_{\pm}(-k,x) = f\pm(k,x) \text{ for } \mathrm{Im}(k) \geq 0 \tag{2.32}$$

and for real k:

$$f_{\pm}(k) = g_{\pm}(-k) = g_{\pm}^*(k) = f_{\pm}^*(-k). \tag{2.33}$$

2.4 Jost Solutions

The functions $f_{\pm}$ and $g_{\pm}$ are known as the Jost solutions of the Schrödinger equation. For a continuous potential the Wronskians $W(g_{\pm}, f_{\pm})$ would be independent of x because both functions are solutions of the Schrödinger equation with the same k^2. It remains to be true in case of a measure-valued potential because $\frac{\partial}{\partial x} W(g_{\pm}, f_{\pm}) = 0$ almost everywhere. Integration from x_1 to x_2 shows that the Wronskians are independent of x. By calculating the limit $x \to \pm\infty$ we get

$$W(g_{\pm}, f_{\pm}) = \pm 2ik \text{ with } k \in \mathbb{R}. \tag{2.34}$$

For $\mathrm{Im}(k) \geq 0$ we define $W(k) := W(f_-(k,x), f_+(k,x))$. The function $W(k)$ is independent of x, analytic for $\mathrm{Im}(k) > 0$ and continuous for $\mathrm{Im}(k) \geq 0$. For real k we have four solutions of a second order differential equation, which can not be linearly independent: We use coefficients $c_\mp$ and $d_\mp$ and the linearly independence of the pairs (f_+, g_+) and (f_-, g_-) to express $f_\pm$ with:

$$f_\pm(k,x) = c_\mp(k) f_\pm(k,x) + d_\mp(k) g_\mp(k,x) k \in \mathbb{R}, k \neq 0. \tag{2.35}$$

The coefficients $c_\mp$ and $d_\mp$ can be written in terms of Wronskians

$$f_\pm g'_\mp = c_\mp f_\mp g'_\mp + d_\mp g_\mp g'_\mp \tag{2.36}$$
$$f'_\pm g_\mp = c_\mp f'_\mp + d_\mp g'_\mp g_\mp \tag{2.37}$$

and hence

$$c_\pm = \mp \frac{W(f_\mp, g_\pm)}{2ik} \tag{2.38}$$
$$d(k) = d_+(k) = d_-(k) = \frac{W(k)}{2ik}. \tag{2.39}$$

The above function $d(k)$ is used as a new definition for the case $\mathrm{Im}(k) \geq 0$, $k \neq 0$, because in (1.28) d will be only defined for real $k \neq 0$. We use the known relations from (1.26) and get

$$d(k)^* = d(-k) \tag{2.40}$$
$$c_\pm(k)^* = c_\pm(-k) \tag{2.41}$$
$$c_+(-k) = -c_-(k). \tag{2.42}$$

Proposition 2. *If μ satisfies property (2.2) with $p = 2$ then the following holds:*

(i) $W(k) \neq 0$ *for* $\mathrm{Im}(k) \leq 0$ *unless* k *is pure imaginary.*

(ii) Suppose $W(k_0 = 0)$ *for some* k_0 *with* $\frac{k_0}{i} > 0$. *Then*

$$\dot{W}(k_0) = 2k_0 \int_{-\infty}^{\infty} f_-(k_0, x) f_+(k_0, x) dx \neq 0$$

i.e. all zeros of W(k) are simple.

(iii) $d(k) = 1 + O(|k|^{-1})$ *as* $|k| \to \infty$

(iv) $c_\pm(k) = O(|k|^{-1})$ *as* $k \to \pm\infty$

(v) The following alternative holds:
.) Either $d(k)$ *is continuous at* $k = 0$ *with* $d(0) \neq 0$ *and* $c_\pm(k)$ *are continuous at* $k = 0$,
.) or $kd(k)$ *is continuous at* $k = 0$ *with* $\lim_{k \to 0} kd(k) = \alpha \neq 0$ *and* $kc_\pm(k)$ *are continuous at* $k = 0$ *with* $\lim_{k \to 0} kc_\pm(k) = \beta_\pm \neq 0$.

Proof. (i):
The identity $W(f,g)W(\phi,\psi) = W(f,\psi)W(\phi,g) - W(f,\phi)W(\psi,g)$ and the relations (2.33) and (2.34) imply for real $k \neq 0$

$$W(f_+,f_-)W(g_+,g_-) = W(f_+,g_-)W(g_+,g_-) - W(f_+,g_+)W(g_-,f_-) = \\ = |W(f_+,g_-)|^2 - (-2ik)^2 \geq 4k^2 > 0. \tag{2.43}$$

Now look at $\mathrm{Im}(k) > 0$ and suppose $W(k) = 0$. Then $\frac{k}{i} > 0$, because otherwise the self-adjoint operator H would have a nonreal eigenvalue.

(ii):
Now suppose k_0 is a zero of W. Differentiate W with respect to k gives

$$\dot{W} = \frac{d}{dk}W = W(\dot{f}_-,f_+) + W(f_-,\dot{f}_+).$$

We differentiate the two Wronskians on the right-hand side with respect to x and the Schrödinger equation with respect to k to get

$$\frac{\partial}{\partial x}W(\dot{f}_-,f_+) = 2kf_+f_-,$$
$$\frac{\partial}{\partial x}W(f_-,\dot{f}_+) = -2kf_+f_-.$$

Now we have, since $f_+(k_0,x) = \alpha f_-(k_0,x)$, for $k = k_0$

$$\lim_{x\to-\infty} W(\dot{f}_-,f_+) = \alpha \lim_{x\to-\infty} W(\dot{f}_-,f_-) = 0$$

since $|W(\dot{f}_-,f_-)| \leq |\dot{f}_-f'_-| + |\dot{f}'_-||f_-| \leq const|x|^2e^{\mathrm{Im}(k)x} \to 0$ using the estimates in Theorem (2.1). Similarly $\lim_{x\to\infty} W(f_-,\dot{f}+)$. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} 2k_0f_+f_-dx &= \int_{-\infty}^{0} \frac{\partial}{\partial x}W(\dot{f}_-,f_+)dx - \int_{0}^{\infty} \frac{\partial}{\partial x}W(f_-,\dot{f}_+) \\ &= W(\dot{f}_-,f_+)(k_0,0) + W(f_-,\dot{f}_+)(k_0,0) = \dot{W}(k_0,0)\end{aligned}$$

and $\int_{-\infty}^{\infty} 2k_0f_+f_-dx = 2k_0\alpha\int_{-\infty}^{\infty} f_-^2dx \neq 0$, since $f_-(k_0,x)$ is real-valued (k_0 pure imaginary).

(iii:)
We note that for $\mathrm{Im}(k) \leq 0$ as x tends to infinity

$$
\begin{aligned}
W(k) &= W(f_-(k,x), f_+(k,x)) \\
&= \left(e^{-ikx} + \frac{1}{k}\int_{-\infty}^{x} \sin(z) f_- d\mu(x')\right)\left(ike^{ikx} - \int_x^\infty \cos(z) f_+ d\mu(x')\right) \\
&\quad - \left(e^{ikx} - \frac{1}{k}\int_x^\infty \sin(z) f_+ d\mu(x')\right)\left(-ike^{-ikx} + \int_{-\infty}^{x} \cos(z) f_- d\mu(x')\right) \\
&= 2ik - e^{ikx}\int_{-\infty}^{x} (\cos(z) - i\sin(z))\, f_- d\mu(x') - f_-(k,x)e^{ikx}\int_x^\infty \cos(z) e^{-iz} e^{-ikx'} f_+ d\mu(x') \\
&\quad + f_-'(k,x)e^{ikx}\frac{1}{k}\int_x^\infty \sin(z) e^{-iz} e^{-ikx'} f_+ d\mu(x') \\
&= 2ik - \int_{-\infty}^{\infty} e^{ikx'} f_-(k,x') d\mu(x') \qquad (2.44)
\end{aligned}
$$

where $z = k(x - x')$ and where the last step uses that $W(k)$ does not depend on x. According to estimate (2.12) the integral in (2.44) is bounded as a function of k. Now we use the fact that $d(k) = \frac{W(k)}{2ik}$ to finish the claim.

(iv):
The proof is analogous to 3, in particular one obtains for $k \in R$

$$
c_\pm(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{\mp ikx'} f_-(\pm k, x') d\mu(x'). \qquad (2.45)
$$

(v):
We define $v = -W(0) = \int_{-\infty}^{\infty} f_-(0,x') d\mu(x')$ and use (2.44) to obtain

$$
d(k) = 1 - \frac{v}{2ik} - \int_{-\infty}^{\infty} \frac{e^{ikx'} f_-(k,x') - f_-(0,x')}{2ik} d\mu(x) \qquad (2.46)
$$

for $\mathrm{Im}(k) \le 0$.
Note that

$$
|\frac{1}{2ik}(e^{ikx'} f_-(k,x') - f_-(0,x'))| \le \sup_{\mathrm{Im}(k)\le 0} |\frac{1}{2}\frac{\partial}{\partial k}(e^{ikx'} f_-(k,x'))| \le const(1 + x'^2)
$$

which implies the existence of the integral in (2.46). The alternative in the statement appears because v can be zero or not. In the case of v being non-zero $kd(k)$ is continuously approaching $-\frac{v}{2i} \ne 0$ for $k \to 0$. In the first case the limit is evaluated for real k by writing (2.43) as

$$
|d(k)|^2 = 1 + |c_-(k)|^2 \ge 1 \qquad (2.47)
$$

by using (2.38) and (2.39) and we get $d(0) \ne 0$. A similar proof leads to the statements on $c_\pm$. □

2.5 Associated Schrödinger Operator

Now we will discuss the Schrödinger operator associated to the measure μ. Let (2.2) be true for $p = 2$ and define

$$H = -\frac{d^2}{dx^2} + d\mu$$

in $D(H) = \{g \in L^2(\mathbb{R}) | g, g' \in AC_{loc}(\mathbb{R}), -g'' + g\ d\mu \in L^2(\mathbb{R})\}$.

Lemma 2. *The operator H is well-defined and self-adjoint in $L^2(\mathbb{R})$. Its spectrum has the following properties:*

(i) $\sigma_{ess}(H) = \sigma_{ac}(H) = [0, \infty)$

(ii) $\sigma_{sing}(H) = \emptyset$

(iii) $\sigma_{pp}(H) \subseteq (-\infty, 0)$

Proof. We want to refer to the more general case of Sturm-Liouville operators which is treated in [5]. □

Theorem 2.2. *The eigenvalues of H are given through the zeros of the function $d(k)$. In particular there are only finite many eigenvalues and* $\inf \sigma(H) > -\infty$*.*

Proof. We suppose k_0 is a zero of d, i.e., $\frac{k_0}{i} > 0$. This would imply $W(k_0) = 0$ and hence f_+ and f_- are linearly dependent. By Theorem (2.1) f_+ is in $L^2(0, \infty)$ and, because it is a multiple of f_-, it is also in $L^2(-\infty, 0)$. The functions $f_+, f_- \in AC_{loc}(\mathbb{R})$ and $-f_+'' + f_+ d\mu = k_0^2 f_+$ which is again in $L_2(\mathbb{R})$. This means f_+ is an eigenfunction of H with eigenvalue $k_0^2 f_+$. If $k/i > 0$ and $d(k) \neq 0$ we see that f_+ grows exponentially at $-\infty$ and for $k/i > 0$ the function f_- grows exponentially at $+\infty$. Therefore k^2 cannot be an eigenvalue of H which shows the claim that the zeros of d give the eigenvalues of the operator H. Now we prove the statement on the exponential growth of f_- near infinity. A similar argument can be used for f_+. We start with $k/i > 0$ and $d(k) \neq 0$. By (2.44)

$$\begin{aligned} e^{ikx} f_-(k, x) &= 1 + \int_{-\infty}^{x} \frac{e^{2ik(x-x')} - 1}{2ik} e^{ikx'} f_-(k, x) d\mu(x') \\ &= d(k) + \frac{1}{2ik} \int_{x}^{\infty} e^{ikx'} f_-(k, x') d\mu(x') + \frac{1}{2ik} \int_{-\infty}^{x} e^{2ik(x-x')} e^{ikx'} f_-(k, x') d\mu(x'). \end{aligned}$$

Theorem 2.1 shows that $|e^{ikx'} f_-(k, x')|$ is bounded. Hence the first integral tends to zero as x tends to infinity. The second integral is split up in two parts:

$$\begin{aligned} & \int_{-\infty}^{x} e^{2ik(x-x')} e^{ikx'} f_-(k, x') d\mu(x') \\ = \quad & e^{ikx} \int_{-\infty}^{x/2} e^{2ik(x/2-x')} e^{ikx'} f_-(k, x') d\mu(x') + \int_{x/2}^{x} e^{2ik(x-x')} e^{ikx'} f_-(k, x') d\mu(x') \end{aligned}$$

where its absolute value is bound by

$$const\left(e^{-\mathrm{Im}(k)x}\int_{-\infty}^{x/2}1d\mu(x')+\int_{x/2}^{x}1d\mu(x')\right)\leq const\left(\|1\|_1 e^{-\mathrm{Im}(k)x}+\int_{x/2}^{x}1d|\mu|(x')\right)$$

which tends to zero as x tends to infinity. We have $e^{ikx}f_-(k,x)=d(k)+o(1)$ or $|f_-(k,x)|=|d(k)+o(1)|e^{\mathrm{Im}(k)x}\geq\frac{1}{2}|d(k)|e^{\mathrm{Im}(k)x}$. By the analyticity of W for $\mathrm{Im}(k)>0$, its zeros can only cluster at zero and infinity. This implies that also d had infinitely many zeros near zero or infinity. Having both is impossible since d is non-zero or even blowing up at zero and is close to one for large $|k|$. Therefore there exist only finitely many isolated eigenvalues of H. □

We found $\sigma_{pp}(H)=\{-\kappa_j^2|\kappa_j>0, j=1,....,N\}$ *is the set of eigenvalues of* H*. The corresponding eigenfunctions to each eigenvalue* κ_j^2 *are*

$$f_+(i\kappa_j,x)=\mu_j f_-(i\kappa_j,x) \tag{2.48}$$

for some non-zero μ_j*. We define norming constants* $\gamma_{\pm,j}, j=1,...,N$ *by*

$$\gamma_{\pm,j}=\|f_\pm(i\kappa_j,.)\|_2^{-1}, j=1,...,N. \tag{2.49}$$

Now we turn to the continuous spectrum and introduce the scattering matrix.

Definition 1. *For real k let*

$$S(k)=\begin{pmatrix}T_-(k) & R_+(k)\\ R_-(k) & T_+(k)\end{pmatrix} \tag{2.50}$$

where $T_-(k)=T_+(k)=T(k)=1/d(k)$ *denote the transmission coefficients with respect to left and right incidence.* $R_\pm(k)=c_\pm(k)/d(k)$ *denote the reflection coefficient with respect to left and right incidence.* $T(k)$ *and* $R_\pm(k)$ *are well defined even if* $k\to 0$ *because of Proposition 2(5). The relations (2.39) imply*

$$T(k)^*=T(-k),\quad R_\pm(k)^*=R_\pm(-k). \tag{2.51}$$

Theorem 2.3. *Assume* μ *satisfies (2.2) with* $p=2$ *and* $k\in R$*. Then* $S(k)$ *is a continuous, unitary operator. In particular,*

$$|R_-(k)|^2 = |R_+(k)|^2 \tag{2.52}$$

$$|T(k)|^2+|R_\pm(k)|^2 = 1. \tag{2.53}$$

Proof. $|R_-(k)|^2=\frac{W(g_-,f_+)W(f_-,g_+)}{W(f_-,f_+)W(f_-,f_+)^*}=|R_+(k)|^2$ because of (2.33). We proof (2.53) by dividing

$$|d(k)|^2=|c_\pm(k)|^2+1$$

by $|d(k)|^2$.

Similarly $c_+(-k) = -c_-(k)$ implies

$$R_+(k)^* T(k) + T(k)^* R_-(k) = 0. \tag{2.54}$$

The unitarity of S is implied by equations (2.53) and (2.54). Continuity follows from continuity of $f_\pm$ and $g_\pm$ and their x-derivatives for $k \neq 0$ and from Proposition 2(5) at $k = 0$. □

Definition 2. *The sets*

$$S_\pm = \{R_\pm(k), k \leq 0; k_j, \gamma_{\pm,j}, j = 1, ..., N\}$$

are called the scattering data $S_\pm$ for H.

The direct scattering step consists in obtaining the scattering data for μ to determine the map

$$\mu \mapsto S_\pm \tag{2.55}$$

All the information in $S_\pm$ is contained in $f_\pm$ and $g_\pm$ and their x-derivatives, which can be obtained by solving the respective Volterra integral equations.

2.6 Physical Interpretation

We want to extract the physical meaning of the scattering matrix. For this purpose we define the wave function of H by

$$\psi_\pm(k) = d(k)^{-1} f_\pm(k, x), k \in \mathbb{R}, x \in \mathbb{R}. \tag{2.56}$$

The expression $H\psi_\pm = k^2\psi\pm$ is meant in the distributional sense. The expansion of $f_\pm$ in terms of f_-, g_- and f_+, g_+, respectively, and the asymptotic behavior of these can be written as

$$\begin{aligned} \psi_+(k, x) &= \begin{cases} d^{-1}(k)e^{ikx} & \text{as } x \to \infty \\ \frac{c_-(k)}{d(k)}e^{-ikx} + e^{ikx} & \text{as } x \to -\infty \end{cases} \\ &= \begin{cases} T(k)e^{ikx} & \text{as } x \to \infty \\ e^{ikx} + R_-(k)e^{-ikx} & \text{as } x \to -\infty. \end{cases} \end{aligned}$$

Similarly

$$\psi_-(k, x) = \begin{cases} T(k)e^{-ikx} & \text{as } x \to \infty \\ e^{-ikx} + R_+(k)e^{ikx} & \text{as } x \to -\infty. \end{cases}$$

For the interpretation consider a plane wave $e^{i(kx-wt)}$ of frequency ω and wave number k. For a given $\omega > 0$ the sign of k gives the direction: For $k > 0$ the wave travels to the right, for $k < 0$ to the left. If k is positive and we look at a constant frame of time e^{ikx} means waves traveling to the right and e^{-ikx} to the left.

The condition that μ satisfies property (2.2) with $p = 2$ needs a decay of the potential at $\pm\infty$, where the solutions are close to plane waves. Then ψ_+ describes a wave coming from $-\infty$. A part of it ($T(k)e^{+ikx}$) will be transmitted to $+\infty$ and another part of it ($R_-(k)e^{-ikx}$) will be reflected back to $-\infty$. Similarly for ψ_+ which travels the other direction. The transmission coefficient is independent from whether the wave comes from $+\infty$ or $-\infty$. For the reflection coefficient the starting point will not be irrelevant, but we have $|R_+| = |R_-|$. This means that the scattering matrix includes the asymptotic information which describes the whole direct scattering process.

3 Inverse Scattering

3.1 Reconstruction of Scattering Data

Theorem 3.1. *The scattering matrix $S(k)$ can be reconstructed from one of the two scattering data S_+ or S_-.*

Proof. We prove the claim for S_+ as the scattering data and define for $Im(k) \geq 0$

$$h(k) = T(k) \prod_{i=1}^{N} \frac{k - i\kappa_j}{k + i\kappa_j}. \tag{3.1}$$

Since $d(k)$ has simple zeros, $T(k)$ has simple poles at $i\kappa_j$ for $j = 1, ..., N$ and these are removed by the factors $(k - i\kappa_j)$. The factors $(k + i\kappa_j)^{-1}$ are analytic for $Im(k) \leq 0$ and therefore $h(k)$ is analytic in the upper half plane and continuous down to the real line. It has no zero in the upper half plane and on the real line, except for $k = 0$. We first suppose $d(0)$ is finite, i.e., $h(k)$ has no zero. This implies $\log h(k)$ is analytic in the upper half plane and it is continuous down to the real line. Let Γ_R be the closed curve of the segment $[-R, R] \subseteq \mathbb{R}$ and C_R a semicircle of radius R connecting R and $-R$ through the upper half plane. According to Theorem I.13.3 of Markushevich [7] Cauchy's integral formula can be generalized to yield

$$\ln h(k) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{\ln h(\xi)}{\xi - k} d\xi \text{ for } \mathrm{Im}(k) > 0. \tag{3.2}$$

Now let $R \geq 2|k|$. Then

$$\begin{aligned} |\int_{C_R} \frac{\ln h(\xi)}{|\xi - k|} d\xi| \leq \pi R \sup_{\xi \in C_R} |\frac{\ln h(\xi)}{\xi - k}| \leq \frac{\pi R}{R - |k|} \sup_{\xi \in C_R} |\ln\ h(\xi)| \\ \leq 2\pi \sup_{\xi \in C_R} \sup\{|\ln |h(\xi)|| + |\arg h(\xi)|\}. \end{aligned} \tag{3.3}$$

As ξ tends to infinity $h(\xi) = 1 + O(\frac{1}{\xi})$. This implies that both $|\ln |h(\xi)||$ and $|\arg\ h(\xi)|$ vanish at the rate $\frac{1}{R}$. Hence

$$\ln h(k) = \lim_{R \to \infty} \left(\frac{1}{2\pi i} \int_{-R}^{R} \frac{\ln\ h(\xi)}{\xi - k} d\xi + r(R, k) \right) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln h(\xi)}{\xi - k} d\xi \tag{3.4}$$

where $r(R, k)$ denotes the contribution of the semicircle C_R and satisfies $|r(R, k)| \geq \frac{const}{R}$. If $h(k)$ has the power series $\sum_{n=0}^{\infty} a_n (k - k_0)^n$ near k_0 for $Im(k) > 0$, then $h(k^*)^*$ is an analytic function for $Im(k) < 0$ and continuous up to the real line. Using now a semicircle through the lower half plane Cauchy's theorem yields for $Im(k) > 0$

$$0 = \frac{1}{2\pi i}\int_{\Gamma_R^*} \frac{\ln\ h(\xi^*)^*}{\xi - k} d\xi = \lim_{R\to\infty}\left(\frac{1}{2\pi i}\int_{-R}^{R} \frac{\ln(h(\xi)^*)}{\xi - k} d\xi + \overline{r}(R,k)\right)$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(h(\xi)^*)}{\xi - k} d\xi \tag{3.5}$$

where $\overline{r}(R,k)$ accounts for contributions of C_R^* and vanishes as R tends to infinity. Adding up (3.4) and (3.5) we obtain

$$\ln h(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln|h(\xi)|^2}{\xi - k} d\xi,\ Im(k) > 0 \tag{3.6}$$

which yields

$$T(k) = \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |R_+(\xi)|^2)}{\xi - k} d\xi\right),\ Im(k) > 0 \tag{3.7}$$

since $|h(\xi)|^2 = |T(\xi)|^2 = 1 - |R_\pm(\xi)|^2$. Next consider the case where $d(k)$ blows up as $k \to \infty$, i.e., where $h(k)$ has a zero at $k = 0$. In this case we circumvent zero on a small semicircle c_ϵ with radius $\epsilon \leq 1/2|k|$ and prove that the contribution of this integral tends to zero as $\epsilon \to 0$. One has

$$|\int_{c_\epsilon} \frac{\ln h(\xi)}{\xi - k} d\xi| \leq \frac{2\pi\epsilon}{|k|} \sup_{\xi\in c_\epsilon} |\ln h(\xi)|. \tag{3.8}$$

But

$$|\ln h(\xi)| = |\sum_{j=1}^{N} \ln\frac{\xi - i\kappa_j}{\xi + i\kappa_j} - \ln\frac{\xi d(\xi)}{\xi}|$$
$$\leq \sum_{j=1}^{N} |\ln\frac{\xi - i\kappa_j}{\xi + i\kappa_j}| + |\ln(\xi d(\xi))| + |\ln(\xi)|.$$

All these terms are bounded, except for $|\ln\xi|$, which is estimated by $|\ln\xi| \leq |\ln(\xi)| + |arg\ \xi| \leq |\ln\epsilon| + \pi$. Then the claim follows since $\epsilon\ln\epsilon \to 0$. Finally for $k \in \mathbb{R}$ we have $T(k) = \lim_{\epsilon\to 0} T(k + i\epsilon)$ since T is continuous down to the real line. This turns (3.7) into

$$T(k) = \sum_{j=0}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |R_+(\xi)|^2)}{\xi - k} d\xi + \frac{1}{2}\ln(1 - |R_+(k)|^2)\right),\ k \in \mathbb{R},$$

where the integral is to be interpreted as Cauchy's principal value, i.e., $\int_{-\infty}^{\infty} ... = \lim_{\epsilon\to 0}(\int_{-\infty}^{k-\epsilon} ... + \int_{k+\epsilon}^{\infty})$ (Markushevich [7], section I.74). So far we have obtained $T(k)$ known $R_+(k), k \geq 0$ (notice that $R_+(-k) = R_+(k)^*$) and for known values of κ_j for $j = 1, ..., N$. To prove the theorem we still have to obtain $R_-(k)$ and $\gamma_{-,j}$ for $j = 1, ..., N$, but R_- is determined using equation

(2.54). Thus the scattering matrix is reconstructed. Concerning the construction of $\gamma_{-,j}$, note that we proved in part 2 of Proposition (2) that $\dot{W}(i\kappa_j) = 2i\kappa_j \int_{-\infty}^{\infty} f_-(i\kappa_j, x) f_+(i\kappa_j, x) dx$, if $-\kappa_j^2$ is an eigenvalue of H. But then $W(i\kappa_j) = 0$ and $f_+(i\kappa_j, x) = \alpha_j f_-(i\kappa_j, x)$. Hence

$$
\begin{aligned}
[\frac{d}{dk}\frac{1}{T(k)}]_{k=i\kappa_j} = [\frac{d}{dk}\frac{W(k)}{2ik}]_{k=i\kappa_j} &= -i\int_{-\infty}^{\infty} f_-(i\kappa_j, x) f_+(i\kappa_j, x) dx \\
&= -i\alpha_j \|f_-\|_2^2 = \frac{-i}{\alpha_j}\|f_+\|_2^2 \\
&= -i\alpha_j \gamma_{-2}^{-2} = -i\alpha_j^{-1}\gamma_{+,j}^{-2}.
\end{aligned}
$$

Therefore, since we know $T(k)$ and $\gamma_{+,j}$ we can calculate α_j and also $\gamma_{-,j}$. Hence we have reconstructed the scattering matrix $S(k)$ and the scattering data S_- from S_+. We can also start with S_- and reconstruct S_+. □

Theorem 3.2. *Consider a continuous function $R_+(k) : \mathbb{R} \to \mathbb{C}$ satisfying the following conditions:*
1. $R_+(k) = R_+(-k)^*$
2. $|R_+(k)| \leq 1$, $|R_+(k)| = 1 \Rightarrow k = 0$
3. If $|R_+(0)| = 1$, *then* $\lim_{k\to 0} \frac{1+R_+(k)}{k} = \rho_+ \neq 0$
4. $|R_+(k)| = O(\frac{1}{|k|})$ *as* $k \to \infty$.
and positiv distinct numbers $\kappa_1, ..., \kappa_N$. *We define the function* $T(k)$ *for* $Im(k) > 0$ *by*

$$T(k) = \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \exp\left(\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\ln(1 - |R_+(\xi)|^2)}{\xi - k} d\xi\right) \tag{3.9}$$

and for real k *by*

$$
\begin{aligned}
&T(k) = \lim_{\epsilon\to 0} T(k + i\epsilon) \\
= \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \exp\left(\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\ln(1 - |R_+(\xi)|^2}{\xi - k} d\xi + \frac{1}{2}\ln(1 - |R_+(k)|^2)\right). &\qquad (3.10)
\end{aligned}
$$

Note that the last integral is a principal value integral. $T(k)$ *is meromorphic in the upper half plane which have only simple poles at* $i\kappa_1, ..., i\kappa_N$. *It is continuous down to the real axis.* $T(k)$ *has the asymptotic behavior* $1 + O(|k|^-1)$ *as* $|k| \to \infty$ *and satisfies* $T(-k) = T(k)^*$ *for real* k. *Furthermore* $|T(k)| > 0$ *except possibly if* $k = 0$. *The behavior at* $k = 0$ *is either* $|T(0)| \neq 0$ *or* $\lim_{k\to\infty} \frac{T(k)}{k} = \alpha \neq 0$. *Next we define* $R_-(k) = -\frac{R_+(k)^* T(k)}{T(k)^*}$ *which satisfies all above mentioned conditions on* R_+ *with* ρ_+ *replaced by a different number* ρ_-. *The matrix*

$$S(k) = \begin{pmatrix} (T(k) & R_+(k) \\ R_-(k) & T(k) \end{pmatrix}, k \in \mathbb{R}$$

is continuous and unitary in particular $|T(k)|^2 + |R_+(k)|^2 = 1$.

Proof. The analyticity in $Im(k) > 0$ of the exponential function in (3.9) is proven by Lemma IV.5.1 of Conway [2] and the fact that the sequence obtained by integrating only over

$[-R,R]$ converges uniformly. Continuity follows by definition. This yields the statement on the meromorphicity and continuity of T. All the factors $\frac{k+i\kappa_j}{k-i\kappa_j}$ have the asymptotic behavior $1+O(|k|^{-1})$ as $|k|\to\infty$. Therefore we consider once more the argument of the exponential function and show that it tends to zero like $\frac{const}{|k|}$ with $|k|\to\infty$. The property $T(k)=T(-k)^*$ for real k is implied by $|R_+(-\xi)|=|R_+(\xi)|$. If $|R_+(0)\neq 1$, then $\ln(1-|R_+(\xi)|^2)$ is finite for all ξ and therefore the argument of the exponential function is, allowing no zero for $T(k)$. On the other hand, if $R_+(\xi)=-1+O(\xi)$ as $\xi\to 0$, then

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\ln(1-|R_+(\xi)|^2)}{\xi-i\epsilon}d\xi &= \int_0^{\infty}\frac{\ln(1-R_+(\xi)R_+(-\xi))}{\xi-i\epsilon}d\xi+\int_{-\infty}^{0}\frac{\ln(1-R_+(\xi)R_+(-\xi))}{\xi-i\epsilon}d\xi\\ &= \int_0^{\infty}\ln(1+|R_+(\xi)|^2)(\frac{1}{-\xi-i\epsilon}+\frac{1}{\xi-i\epsilon})d\xi\\ &= i\epsilon\int_{-\infty}^{\infty}\frac{\ln(1-|R_+(\xi)R_+(-\xi)|)}{\xi^2+\epsilon^2}d\xi \qquad (3.11)\end{aligned}$$

This last expression tends to infinity like $|\ln\epsilon|$ as ϵ tents to zero, since $1-|R_+(\xi)|^2\geq const\ \xi^2$. $T(i\epsilon)$ behaves like $\alpha\epsilon+o(\epsilon)$ for some $\alpha\neq 0$. $R_-(k)=R_-(-k)^*$ is immediate from the corresponding properties for R_+ and T. Since $|R_-(k)|=|R_+(k)|$ there is only need to fix the phase of R_- at zero in the case that $|R_+(0)|=|R_-(0)|=1$. Nothing that $\lim_{k\to 0}\frac{T(k)}{T(-k)}=-1$ one gets

$$0=1+R_+(k)+O(k)=1-R_-(k)^*\frac{T(k)}{T(-k)}+O(k)=1+R_-(k)^*+o(1),\text{ i.e., } R_-(0)=-1$$

It remains to show that $|T(k)|^2+|R_+(k)|^2=1$. But $|e^{x+iy}|=e^x$ and $|\frac{z^*}{z}|=1$ yields $|T(k)|=\exp(\frac{1}{2}\ln(1-|R_+(k)|^2))$, which is the desired result. □

We will now introduce some results on Fourier transforms and Hardy functions:

Definition 3. *A function h is called Hardy function, if it is analytic in the upper halfe plane and satisfies*

$$\sup_{b>0}\int_{-\infty}^{\infty}|h(a+ib)|^2da<\infty.$$

Lemma 3. *A function h is of class H^{2+} if and only if for all $Im(\omega)>0$ h can be written as*

$$h(\omega)=\int_0^{\infty}f(x)e^{i\omega x}dx$$

for some $f\in L^2(\mathbb{R})$ vanishing on the negative real axis. For a proof see [3] (Thm 3.4.1).

Now we have a look at some facts about Fourier transform. The Fourier transform of f is denoted by $f^\wedge$ and defined by

$$f^\wedge(y)=\int_{-\infty}^{\infty}f(x)e^{ixy}dx. \qquad (3.12)$$

The inverse transform of f is denoted by $f^\vee$ and defined by

$$f^\vee(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-ixy}dy. \tag{3.13}$$

Here we recall some information about Fourier transform. Transform and its inverse are bijective maps from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. Therefore we can write $(f^\vee)^\wedge = (f^\wedge)^\vee = f$. They are not compulsorily unitary but with Plancherel identity we can write $\|f^\vee\|_2 = \sqrt[2]{2\pi}\|f\|_2$. The integral in (3.12) and (3.13) are not necessarily Lebesgue integrals, but this can be fixed by the way of its construction. The transform and its inverse transformation can be defined by a limiting procedure. We define the transforms on a dense subset of $L^2(\mathbb{R})$, where the integrals make sense as Lebesgue integrals. As an example we can use the space of functions of rapid decrease $S(\mathbb{R})$. Then for a function $f \in L^2(\mathbb{R})$ choose a sequence $f_n \in S(\mathbb{R})$ such that $\|f_n - f\|_2 \to 0$. The transforms $f_n^\vee$ of f_n form a Cauchy sequence with respect to the norm in $L^2(\mathbb{R})$. They converge in $L^2(\mathbb{R})$ to a function $f^\vee$ which can be defined to be the transform of f. Note that $f^\vee$ is independent of the chosen sequence. A similar procedure works for the inverse transform.

This all works for the space of square integrable functions, but for $L^1(\mathbb{R})$ the situation is very different. Equation (3.12) defines the Fourier transform for each element of $L^1(\mathbb{R})$ as an ordinary Lebesgue integral. The function $f^\vee(y)$ is bounded and continuous and tends to zero as $|y| \to \infty$. But it need not to be integrable and therefore an inverse transform can not be defined easily. We summarize some facts about Fourier transforms in the special case of $L^1(\mathbb{R})$ in the following:

1) If $f^\vee$ is actually in $L^1(\mathbb{R})$, then f can be recovered by applying the inverse transform (3.13).

2) In the general case f can be recovered from $f^\vee$ using the following recipe. Define the functions f_t as the inverse transformation of $e^{-\frac{1}{2}y^2 t}f^\vee(y)$, for $t > 0$, i.e.,

$$f_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y^2t/2}f^\vee(y)e^{-ixy}dy.$$

Then $\lim_{t\to 0}\|f_t - f\| = 0$.

3) The convolution of $f, g \in L^1(\mathbb{R})$ is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

and is again in $L^1(\mathbb{R})$ by Young's inequality and its Fourier transform is given by

$$(f * g)^\vee = f^\vee g^\vee.$$

4) On the other hand, if $f^\vee, g^\vee \in L^2$, then $f^\vee g^\vee \in L^1$ and their formal inverse transform

exists as a Lebesgue integral

$$\begin{aligned}(f^{\vee}g^{\vee})^{\wedge}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^{\vee}(y)g^{\vee}(y)e^{-ixy}dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}f^{\vee}(y)\int_{-\infty}^{\infty} g(\beta)e^{i\beta y}d\beta dy \\ &= \int_{-\infty}^{\infty} g(\beta)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iy(x-\beta)}f^{\vee}(y)dyd\beta \\ &= \int_{-\infty}^{\infty} g(\beta)(f^{\vee})^{\wedge}(x-\beta)d\beta \\ &= (g*f)(x) = (f*g)(x). \end{aligned} \tag{3.14}$$

(For more information we want to refer to [12]) This shows that $h(\omega)$ with $Im(\omega) = 0$ is a function in $L^2(\mathbb{R})$, since it is the Fourier transform of a function $f \in L^2(\mathbb{R})$, i.e., Hardy function assume boundary values $h_0(a) = \lim_{\epsilon\searrow 0} h(a+i\epsilon)$ such that $h_0 \in L^2(\mathbb{R})$.
For $Im(k) \leq 0$ and $x \in \mathbb{R}$ we define the functions

$$\begin{aligned} m_{\pm}(k,x) &= e^{\mp ikx}f_{\pm}(k,x), \\ n_{\pm}(k,x) = T(k)e^{\pm ikx}f_{\mp}(k,x) &= T(k)m_{\mp}(k,x), \\ N_{\pm}(k,x) &= n_{\pm}(k,x) - 1 - \sum_{j=1}^{N}\frac{A_{\pm,j}(x)}{k-i\kappa_j} \end{aligned}$$

where

$$A_{\pm,j(x)} = i\mu_j^{\pm 1}\gamma_{\pm,j}^2 m_{\mp}(i\kappa_j,x) = i\gamma_{\pm,j}^2 e^{\mp 2\kappa_j x}m_{\pm}(i\kappa_j,x).$$

Proposition 3. *The functions* $(m_{\pm}(k,x)-1)$ *and* $N_{\pm}(k,x)$ *are Hardy functions for each* $x \in \mathbb{R}$.

Proof. According to Theorem (2.1) $m_{\pm}$ is analytic in the upper half plane and continuous down to the real axis. By estimate (2.11) we have for $k = a+ib$ and $|a| > 1$

$$|m_{\pm}(k,x) - 1| \leq \frac{const}{\sqrt{a^2+b^2}} \leq \frac{const}{|a|}.$$

Hence $|m_{\pm}(a+ib,x) - 1|$ is square integrable with respect to a and the result may be estimated independently of b. Therefore $(m\pm(k,x)-1)$ is in H^{2+} for each $x \in \mathbb{R}$. The function $n_{\pm}$ has a simple pole, where $d(k)$ has simple zeros, i.e., at the points $i\kappa_j, j = 1,...,N$. The behavior of $n_{\pm}$ near $i\kappa_j$ is

$$n_{\pm}(k,x) = \frac{m_{\mp}(i\kappa_j,x)}{d(i\kappa_j)(k-i\kappa_j)} + O(1).$$

But $\dot{d}(i\kappa_j) = -i\mu_j^{\mp 1}\gamma_{\pm,j}^{-2}$. Hence all the poles are removed in $N_{\pm}(k,x)$.

By Minkowski's inequality

$$\begin{aligned}(\int_{-\infty}^{\infty} |N_\pm(a+ib,x)|^2 da)^{\frac{1}{2}} \\ \leq \left(\int_{-\infty}^{\infty} |N_\pm(a+ib,x)\xi_{[-1,1]}(a)|^2 da\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{\infty} |N_\pm(a+ib,x)\xi_{\overline{[-1,1]}}|^2 da\right)^{\frac{1}{2}} \end{aligned} \tag{3.15}$$

where ξ denotes the characteristic function. Note that $|A_j(x)/(k-i\kappa_j)|$ and $|n_\pm(k,x)-1|$ and hence $|N_\pm(k,x)-1|$ are of order $1/|k|$ as $|k|\to\infty$, since

$$\begin{aligned}|n_\pm(k,x)-1| = |T(k)(m_\mp(k,x)-1)+T(k)-1| \leq |T(k)||m_\mp(k,x)-1|+|T(k)-1| \\ \leq |m_\mp(k,x)-1| + |T(k)||d(k)-1| \leq \frac{const}{|k|}.\end{aligned}$$

Therefore the integrand in the first integral on the right-hand side of (3.15) may be estimated by a constant independent of b. Integration over the finite interval $[-1,1]$ yields to a constant. Also for the second integral the integrand may be estimated by $\frac{const}{|a|^2}$ independently of b. Integration over $(-\infty,-1)$ and $(1,\infty)$ yields to constants. This shows that $N_\pm(k,x) \in H^{2+}$ for each $x \in \mathbb{R}$.

Corollary 2. *There exist functions $B_\pm(x,.)$ and $\tilde{B}_\pm(x,.) \in L^2(\mathbb{R}), x \in \mathbb{R}$ such that $B_\pm(x,y) = 0, \tilde{B}_\pm(x,y) = 0$ for $y < 0$ and all $x \in \mathbb{R}$. We have*

$$\begin{aligned}m_\pm(k,x) &= 1 + \int_{-\infty}^{\infty} B_\pm(x,y)e^{iky}dy,\ \mathrm{Im}(k) \geq 0, x \in \mathbb{R} \\ n_\pm(k,x) &= 1 + \sum_{j=1}^{N} \frac{A_{\pm j}(x)}{k-i\kappa_j} + \int_{-\infty}^{\infty} \tilde{B}_\pm(x,y)e^{iky}dy,\ \mathrm{Im}(k) \geq 0, x \in \mathbb{R}.\end{aligned} \tag{3.16}$$

Furthermore for each x and real argument the function $m_\pm(.,x)-1$ is the Fourier transform of a certain function $b_\pm(x,.) \in L^2(\mathbb{R})$ vanishing on the negative real axis. Similary $N_\pm(.,x)$ is the Fourier transform of a certain function $\tilde{B} \in L^2(\mathbb{R})$ vanishing on the negative real axis.
Using (2.33) we obtain from the fundamental relation

$$f_\pm(k,x) = c_\mp(k)f_\mp(k,x) + d(k)g_\mp(k,x)$$

the equation

$$n_\pm(k,x) = R_\pm(k)e^{\pm 2ikx}m_\pm(k,x) + m_\pm(-k,x) \tag{3.17}$$

or

$$N_\pm(k,x) = -\sum_{j=1}^{N} \frac{A_{\pm j}(x)}{k-i\kappa_j} + (m_\pm(-k,x)-1) + R_\pm(k)e^{2ikx} + R_\pm(k)e^{\pm 2ikx}(m_\pm(k,x)-1).$$

Now we take inverse Fourier transforms on both sides and evaluate at $\mp 2y$. We have for the *left-hand side*

$$(N_{\pm}(.,x)^{\wedge})(\mp 2y) = \tilde{B}_{\pm}(x,\mp 2y),$$

and for the first term on the right-hand side

$$\begin{aligned} -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A_{\pm,j}(x)}{k-i\kappa_j}e^{ik(\mp 2y)}dk &= -iA_{\pm,j}(x)\int_{-\infty}^{\infty}\frac{e^{ik(\pm 2y)}}{2\pi i(k-i\kappa_j)}dk \\ &= -iA_{\pm,j}(x)\begin{cases}\exp(\mp 2\kappa_j y) & \text{for } \pm 2y>0\\ 0 & \text{for } \pm 2y<0\end{cases} \\ &= \begin{cases}\gamma_{\pm j}^2 e^{\mp 2\kappa_j(x+y)}m_{\pm}(i\kappa_j,x) & \text{for } \pm 2y>0\\ 0 & \text{for } \pm 2y<0.\end{cases}\end{aligned}$$

Moreover $(m_{\pm}(-k,x)-1)^{\wedge}(\mp 2y) = B_{\pm}(x,\pm 2y)$. *Next define* $R_{\pm}^{\wedge}(z) = C_{\pm}(\mp\frac{1}{2}z)$, *which yields to*

$$(R_{\pm}(.)e^{\pm 2ix\cdot})^{\wedge}(z) = C_{\pm}(x\mp\frac{z}{2}). \tag{3.18}$$

Therefore the third term on the right hand side becomes $C_{\pm}(x+y)$. *The last term finally is a product of two Fourier transforms:* $C_{\pm}(x,\pm\frac{\cdot}{2})^{\vee}(k)B_{\pm}^{\wedge}(x,.)$ *which has, according to (3.14), the inverse transform*

$$\int_{-\infty}^{\infty}B_{\pm}(x,z)C_{\pm}(x\mp\frac{1}{2}(\mp 2y-z)dz = 2\int_{-\infty}^{\infty}B_{\pm}(x,\pm 2z)C_{\pm}(x+y+z)dz.$$

Collecting all the terms one arrives at

$$\begin{aligned}\tilde{B}(x,\pm 2y) = \sum_{j=1}^{N}\gamma_{\pm,j}^2 e^{\mp 2\kappa_j(x+y)}m_{\pm}(i\kappa_j,x)\chi_{\mathbb{R}^+}(\pm y) + B_{\pm}(x,\pm 2y)\\ +C_{\pm}(x+y) + 2\int_{-\infty}^{\infty}B_{\pm}(x,\pm 2z)C_{\pm}(x+y+z)dz\end{aligned} \tag{3.19}$$

Next note that $\tilde{B}_{\pm}(x,\mp 2y) = 0$ *for* $\pm y > 0$. *The first term on the right hand side of (3.19) can be written in the following way*

$$\begin{aligned}&\sum_{j=1}^{N}\gamma_{\pm j}^2 e^{\mp 2\kappa_j(x+y)}\left(1+\int_{-\infty}^{\infty}B_{\pm}(x,z)e^{-\kappa_j z}dz\right)\\ &= \sum_{j=1}^{N}\gamma_{\pm,j}^2\left(e^{\mp 2\kappa_j(x+y)} + 2\int_{-\infty}^{\infty}B_{\pm}(x,\pm 2z)e^{\mp 2\kappa_j(x+y+z)}dz\right).\end{aligned}$$

Therefore, letting

$$\omega_{\pm}(z) = \sum_{j=1}^{N}\gamma_{\pm,j}^2 e^{\mp 2\kappa_j z} + C_{\pm}(z) = \sum_{j=1}^{N}\gamma_{\pm j}^2 e^{\mp 2\kappa_j z} + \frac{1}{2\pi}\int_{-\infty}^{\infty}R_{\pm}(k)e^{\pm 2ikz}dk \tag{3.20}$$

we get form (3.19) for $\pm y > 0$

$$0 = \omega_\pm(x+y) + B_\pm(x, \pm 2y) + 2\int_{-\infty}^{\infty} B_\pm(x, \pm 2z)\omega_\pm(x+y+z)dz. \tag{3.21}$$

□

3.2 Marchenko Equation

So for a given set of scattering data both ω_+ and ω_- are given functions. If we have one, the other set can be calculated by Theorem (3.1). Equation (3.21) is called Marchenko equation and it is an integral equation, which can be used to obtain $B_\pm(x,y)$ from a given set of scattering data.

We can summarize it to the following theorem.

Theorem 3.3. *The function* $B_\pm(x,y)$ *defined in Corollary (2) satisfies the Marchenko equation*

$$B_\pm(x, \pm 2y) + \omega_\pm(x+y) + 2\int_{-\infty}^{\infty} B_\pm(x, \pm 2z)\omega_\pm(x+y+z)dz = 0, \textit{ for } \pm y > 0$$

where

$$\omega_\pm(z) = \sum_{j=1}^{N} \gamma_{\pm j}^2 e^{\mp 2\kappa_j z} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R_\pm(k) e^{2ikz} dk. \tag{3.22}$$

We take a look at the case of B_- where

$$\begin{aligned}
\int_0^\infty B_-(x,y)e^{iky}dy &= m_-(k,x) - 1 = \int_{-\infty}^{x} \frac{\sin k(x-x')}{k} e^{ik(x-x')} m_-(k,x')d\mu(x') \\
&= \int_{-\infty}^{x} \frac{\sin(k(x-x'))}{k} e^{ik(x-x')}\left(1 + \int_0^\infty B_-(x',z)e^{ikz}dzd\mu(x')\right).
\end{aligned}$$

Now we use that $\frac{\sin(k(x-x'))}{k}e^{ik(x-x')} = \int_0^{x-x'} e^{2ikt}dt$ and therefore

$$\begin{aligned}
B_-(x,.)^\wedge(k) = \int_{-\infty}^{\infty} d\mu(x')\Theta(x-x') \int_{-\infty}^{\infty} dt\Theta(t)\Theta(x-x'-t)e^{2ikt} \\
+ \int_{-\infty}^{\infty} d\mu(x')\Theta(x-x') \int_{-\infty}^{\infty} dt\Theta(x-x'-t) \int_0^\infty dz e^{2ikt} B(x',z)e^{ikz}.
\end{aligned} \tag{3.23}$$

The function Θ denotes the unit step function. We take a look at the first term of the right hand side of the equation and get

$$\begin{aligned}
\int_{-\infty}^{\infty} dt e^{2ikt}\Theta(t) \int_{-\infty}^{x-t} 1 d\mu(x') &= \frac{1}{2}\int_{-\infty}^{\infty} dt e^{ikt}\Theta(t) \int_{-\infty}^{x-t/2} 1 d\mu(x') \\
&= \frac{1}{2}\left(\Theta(.) \int_{-\infty}^{x-./2} d\mu(x')\right)^\wedge (k).
\end{aligned}$$

For the second term we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt \int_{0}^{\infty} dz \int_{-\infty}^{x-t} d\mu(x')\Theta(t)e^{ik(2t+z)}B_-(x',z) \\
= &\int_{-\infty}^{\infty} dy \frac{1}{2} \int_{0}^{\infty} dz e^{iky}\Theta\left(\frac{y-z}{2}\right) \int_{-\infty}^{x-(y-z)/2} d\mu(x')B_-(x',z). \\
= &\frac{1}{2}\left(\int_{0}^{\cdot} dz \int_{-\infty}^{x+(z-.)/2} d\mu(x')B_-(x'z)\right)^{\wedge}(k).
\end{aligned}$$

We take the inverse transformation and see that B_- has to satisfy the integral equation

$$2B(x,y) = \int_{-\infty}^{x-y/2} 1d\mu(x') + \int_{0}^{y}\int_{-\infty}^{x+(z-y)/2} B(x',z)d\mu(x')dz \tag{3.24}$$

for $y > 0$. For $y < 0$ we have no nonhomogeneous term and this implies $B_-(x,y) = 0$.

For the integral equation (3.24) we have the following theorem:

Theorem 3.4. *If μ satisfies property (2.2) with $p = 2$ then the integral equation (3.24) has a (unique) solution $B(x,.) : \mathbb{R}_+ \to \mathbb{R}$ satisfying*

(i) $\|B(x,.)\|_\infty \leq \frac{1}{2}\eta(x)e^{\gamma}(x) < \infty$

(ii) $\|B(x,.)\|_1 \leq \gamma(x)e^{\gamma}(x) < \infty$

(iii) The function $e^{-ikx}(1 + \int_0^\infty B(x,y)e^{iky}dy)$ *is the Jost solution* $f_-(k,x)$

for $\gamma(x) = \int_{-\infty}^{x}(x-t)d|\mu|(t)$ *and* $\eta(x) = \int_{-\infty}^{x} 1d|\mu|(t)$.

The fact that $B(x,.) \in L^\infty(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ implies $B(x,.) \in L^2(\mathbb{R}^+)$ and its Fourier transform exists in L^2. The B in the Jost solution $f_-(k,x) = e^{-ikx}(1 + \int_0^\infty B(x,y)e^{iky}dy)$ is the B_- which was defined in Corollary (2). This is the justification for dealing with (3.24) in the first place.

Proof. We suppose $2B(x,y) = \sum_{n=0}^{\infty} K_n(x,y)$ defined by

$$\begin{aligned}
K_0(x,y) &= \int_{-\infty}^{x-y/2} 1d\mu(x') \\
K_{n+1}(x,y) &= \int_{0}^{y}\int_{-\infty}^{x+(z-y)/2} \frac{1}{2}K_n(x',z)d\mu(x')dz.
\end{aligned}$$

The sum $\sum_{n=0}^{\infty} K_n(x,y)$ solves (3.24) formally and is well defined because it is absolutely convergent. We show this by proving that $K_n(x,y)$ is bounded by $\eta(x-\frac{y}{2})\frac{1}{n!}(\gamma(x))^2$. This is true for $n = 0$ and we use induction for showing the general case. We suppose that it is true for n and show it for $(n+1)$ by

$$|K_{n+1}(x,y)| \leq \int_0^y dz \int_{-\infty}^{x+(z-y)/2} d|\mu|(x')\frac{1}{2}\eta\left(x'-\frac{z}{2}\right)\frac{1}{n!}(\gamma(x'))^n$$
$$\leq \eta(x-\frac{y}{2})\int_0^y dz \int_{-\infty}^{x+(z-y)/2} d|\mu|(x')\frac{1}{2}\frac{1}{n!}(\gamma(x'))^n$$
$$= \frac{1}{2n!}\eta\left(x-\frac{y}{2}\right)\left(\int_{x-\frac{y}{2}}^{x} d|\mu|(x')\gamma^n \int_{2(x'-x)+y}^{y} dz + \int_{-\infty}^{x-\frac{y}{2}} d|\mu|(x')\gamma^n \int_0^y dz\right) \tag{3.25}$$

We further have

$$\begin{aligned} |K_{n+1}(x,y)| &\leq \frac{1}{2n!}\eta\left(x-\frac{y}{2}\right)\left(2\int_{x-y/2}^{x}(\gamma(x'))^n(x-x')d|\mu|(x') + y\int_{-\infty}^{x-y/2}(\gamma(x'))^n d|\mu|(x')\right) \\ &\leq \frac{1}{n!}\eta(x-y/2)\int_{-\infty}^{x}(x-x')(\gamma(x'))^n d|\mu|(x'). \end{aligned}$$

We used that $2x' \leq 2x-y$ and $y \leq 2(x-x')$. For $x' \leq x$ we have

$$\gamma(x') = \int_{-\infty}^{x'}(x'-t)d|\mu|(t) \leq \int_{-\infty}^{x'}(x-t)d|\mu|(t) = \overline{\gamma}(x').$$

Because of $\overline{\gamma}(x) = \gamma(x)$ we get

$$|K_{n+1}(x,y)| \leq \frac{1}{n!}\eta(x-y/2)\int_{-\infty}^{x}\overline{\eta}'(\overline{\eta})^n dx = \frac{1}{(n+1)}\eta(x-y/2)(\eta(x))^{n+1} \tag{3.26}$$

and therefore $\sum_{n=0}^{\infty}|K_n(x,y)| \leq \eta(x-y/2)e^{\gamma(x)} \leq \eta(x)e^{\gamma(x)}$. Every K_n is real so B is real. The L^∞-bound on $B(x,.)$ is clear and we can get the L^1-bound by

$$\int_0^\infty |B(x,y)|dy \leq \frac{1}{2}e^{\gamma(x)}\int_0^\infty \eta(x-y/2)dy = e^{\gamma(x)}\int_{-\infty}^{x}\eta(z)dz = \gamma(x)e^{\gamma(x)},$$

with $\gamma'(x) = \eta(x)$.

□

We suppose a given set of scattering data S_- and define the function $\omega_-(z)$ with the help of (3.21). Hence the homogeneous term and the kernel of the Marchenko equation is defined. Note that the Marchenko equation is a Fredholm integral equation. Under certain conditions on S_- the Marchenko equation has an unique solution $B_-(x,y)$. Now we take a look at the conditions under which the Marchenko equation has an unique solution.

Lemma 4. *The Fredholm Alternative*

Either the inhomogeneous integral equation

$$\phi(x) = f(x) + \int_0^\infty K(x,y)\phi(y)dy \tag{3.27}$$

with L^2 Kernel $K(x,y)$ has an unique L^2-solution ϕ for any given L^2-function$f$ or the homo-

geneous equation

$$\phi(x) = \int_0^\infty K(x,y)\phi(y)dy \tag{3.28}$$

has non trivial L^2-solutions.

For details see Cochran [1].

We have to rewrite the Marchenko equation to apply this lemma for B_- as

$$B_-(x,2y) = -\omega_-(x-y) - 2\int_0^\infty B_-(x,2z)\omega_-(x-y-z)dz,\ y>0. \tag{3.29}$$

The requirements are that $-\omega_-(x-y) \in L^2(\mathbb{R}^+)$ as a function of y and the kernel $-2\omega_-(x-y-z)$ is in $L^2(\mathbb{R}^+ \times \mathbb{R}^+)$ as a function of y and z.

Theorem 3.5. *For distinct positive numbers k_j with $j = 1,...,N$, positive numbers $\gamma_{-,j}$ for $j = 1,...,N$ and a function $R_-(k)$ for $k \in \mathbb{R}$ satisfying the condition of Theorem 3.2 and having an absolutely continuous inverse Fourier transform $(R_-^\vee)(.)$ with $(1+|.|)(R_-^\vee)' \in L^1(\mathbb{R})$ the function ω defined by (3.20) is such that the Marchenko equation has a unique solution.*

Proof. First we prove that $\int_0^\infty |\omega_-(x-y)|^2 dy < \infty$ for each $x \in \mathbb{R}$. Remember that $\omega_-(z) = \sum_{j=1}^N \gamma_{-j}^2 e^{2\kappa_j z + R_-^\vee(2z)}$. Then

$$\int_0^\infty |\omega_-(x-y)|^2 dy \le 2\sum_{j=1}^N \gamma_{-j}^4 e^{4\kappa_j x}\frac{1}{4\kappa_j} + 2\int_0^\infty |R_-^\vee(2x-2y)|^2 dy$$

by (3.18). Since $R_- \in L^2(\mathbb{R})$ one has $R_-^\vee \in L^2$ and the claim follows for each $x \in \mathbb{R}$.

Next we define $p(x) = \int_{-\infty}^x |\frac{d}{dz}|dz$ which is well defined because of $(R_-^\vee)' \in L^1(\mathbb{R})$ and monotone increasing with the property $|\omega_-(x)| \le p(x)$. We get

$$\int_0^\infty dz \int_0^\infty dy|\omega_-(x-y-z)|^2 \le \int_0^\infty dz \int_0^\infty dy\ p(x-y-z)^2$$
$$\le \int_0^\infty dz\ p(x-y)\int_0^\infty dy\ p(x-y-z) = \frac{1}{2}\left(\int_0^\infty p(x-y)dy\right)^2 = \frac{1}{2}\left(\int_{-\infty}^x p(y)dy\right)^2$$

and have to show that $p \in L^1(-\infty, x]$. For $x < 0$

$$p(x) = \int_{-\infty}^x (1+|y|)|\omega_-'(y)|\frac{1}{1+|y|}dy \le \frac{1}{1+|x|}\int_{-\infty}^x (1+|y|)|\omega_-'(y)|dy.$$

Therefore $\lim_{x\to-\infty} xp(x) = \lim_{x\to-\infty} \frac{x}{1+|x|}\int_{-\infty}^x (1+|y|)|\omega_-'(y)|dy = 0$ and

$$\int_{-\infty}^x p(y)dy = yp(y)|_{-\infty}^x - \int_{-\infty}^x y|\omega_-'(y)|dy = xp(x) - \int_{-\infty}^x y|\omega_-'(y)|dy$$

which is finite for $x < \infty$. Now Lemma (4) applies to the Marchenko equation. If the solutions were not unique, then 1 would be a characteristic value and the equation

$$B_-(x,2y) = -2\int_0^\infty B_-(x,2z)w_-(x-y-z)dz \ y>0$$

has a non-trivial solution. Because of $\omega(z) = \omega(z)^*$ we find that B_- is real and

$$\begin{aligned}
0 &< \int_{-\infty}^{\infty} B_-(x,2y)^2 dy = -2\int_{-\infty}^{\infty} B_-(x,2y)\int_0^\infty B_-(x,2z)w_-(x-y-z)dzdy \\
&= -2\int_{-\infty}^{\infty} dy\int_{-\infty}^{\infty} dz B_-(x,2y)B_-(x,2y)\left(\sum_{j=1}^N \gamma_{-,j}^2 e^{2\kappa_j(x-y-z)} + (R_-^\vee)(2x-2y-2z)\right) \\
&= -2\left(\sum_{j=1}^N \gamma_{-j}^2 e^{2\kappa_j} x(\int_{-\infty}^{\infty} B_-(x,2y)e^{-2\kappa_j y}dy\right)^2 \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\int_{-\infty}^{\infty} dz\int_{-\infty}^{\infty} dk B_-(x,2y)B_-(x,2z)R_-(k)e^{-ik(2x-2y-2z)}) \\
&= -2\left(\sum_{j=1}^N \gamma_{-j}^2 e^{2\kappa_j x}\frac{1}{4}(B_-(x,.)^\wedge(i\kappa_j))^2 + \frac{e^{-2ikx}}{8\pi}\int_{-\infty}^{\infty} dk R_-(k)B_-(x,.)^\wedge(k))^2\right). \qquad (3.30)
\end{aligned}$$

Note that $R_-(.)$, $B_-(x,.)$ are L^1-functions and we can use Fubini's theorem in the last step. By the Plancherel identity $\int |B_-(x,2y)|^2 dy = \frac{1}{2}\|B_-(x,.)\|_2^2 = \frac{1}{4\pi}\|B_-(x,.)^\wedge\|_2^2$. We can suppress the x-dependency in $B_-^\wedge$,

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(|B_-^\wedge(k)|^2 + R_-(k)e^{-2ik}(B_-^\wedge(k))^2\right)dk + \frac{1}{2}\sum_{j=1}^N \gamma_{-j}^2 e^{2\kappa_j x}(B_-^\wedge(i\kappa_j))^2 = 0. \qquad (3.31)$$

By introducing

$$\begin{aligned}
f(k) &= B_-^\wedge(-k) + R_-(k)e^{-2ikx}B_-^\wedge(k) \\
g(k) &= T(k)B_-^\wedge(k) \qquad (3.32)
\end{aligned}$$

and using $B_-^\wedge(-k) = B_-^\wedge(k)^*$ and $|T(k)|^2 + |R_-(k)|^2 = 1$ we obtain

$$2\int_{-\infty}^{\infty}(|B_-^\wedge(k)|^2 + R_-(k)e^{-2ikx}(B_-^\wedge(k))^2)dk = \int_{-\infty}^{\infty}(|f(k)|^2 + |g(k)|^2)dk. \qquad (3.33)$$

The expression (3.31) can be written as

$$\frac{1}{4}\int_{-\infty}^{\infty}(|f(k)|^2 + |g(k)|^2 dk + \sum_{j=1}^N \gamma_{-j}^2 e^{2\kappa_j x}(B_-^\wedge(i\kappa_j))^2 = 0$$

and we conclude that $f = 0$ and $g = 0$ almost everywhere since $B_-^\wedge(i\kappa_j)$ is real. This implies $B_-^\wedge = 0$ almost everywhere and therefore $B_-(x,.) = 0$, i.e., B_- is the trivial solution and the uniqueness of the solution of the Marchenko equation is shown. □

The results on inverse scattering could be summarized in the following theorem:

Theorem 3.6. *Suppose the set* $S_- = \{R_-(k), k \in \mathbb{R}; \kappa_1, ..., \kappa_N; \gamma_{-,1}, ..., \gamma_{-,N}\}$ *satisfies the following conditions:*

(i) The numbers $\kappa_1, ..., \kappa_N$ *and* $\gamma_{-,1}, ..., \gamma_{-,N}$ *are positive. The numbers* κ_j *are distinct.*

(ii) The function $R_-(k)$ *is continuous and satisfies:*

- *a)* $R_-(K) = R_-(-k)^*$
- *b)* $|R_-(k)| \leq 1, |R_-(k)| = 1$ *implies* $k = 0$
- *c) If* $|R_-(0)| = 1$, *then* $\lim_{k\to 0} \frac{1+R_-(k)}{k} = \rho \neq 0$
- *d)* $|R_-(k)| = O(\frac{1}{|k|})$ *as* $|k| \to \infty$
- *e)* $R_-^\vee(.)$, *the inverse Fourier transform of* R_-, *is absolutely continuous and* $\int_{-\infty}^{\infty}(1+|x|)|\frac{d}{dx}R_-^\vee(x)|dx < \infty$.

Then S_- *determines* $B(x, y)$ *which is the unique solution of the Marchenko equation*

$$B(x, 2y) + \omega(x-y) + 2\int_0^\infty B(x, 2z)\omega(x-y-z)dz = 0,\ y \geq 0$$

and where $\omega(z) = \sum_{j=1}^N \gamma_{-,j}^2 e^{2\kappa_j z} + R_-^\vee(2z)$.

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