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Almost Everything You Always Wanted to Know About the Toda Equation

Gerald Teschl, Wien

Abstract

The present article reviews methods from spectral theory and algebraic geometry for finding explicit solutions of the Toda equation, namely for the N-soliton solution and quasi-periodic solutions. Along they way basic concepts like Lax pairs, associated hierarchies, and Bäcklund transformations for the Toda equation are introduced.

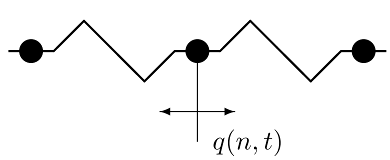
Preface

This article is supposed to give an introduction to some aspects of completely integrable nonlinear wave equations and soliton mathematics using one example, the Toda equation. Moreover, the aim is not to give a complete overview, even for this single equation. Rather I will focus on only two methods (reflecting my personal bias) and I will try to give an outline on how explicit solutions can be obtained. More details and many more references can be found in the monographs by Gesztesy and Holden [19], myself [39], and Toda [40].

The contents constitutes an extended version of my talk given at the joint annual meeting of the Österreichische Mathematische Gesellschaft and the Deutsche Mathematiker-Vereinigung in September 2001, Vienna, Austria.

1 The Toda equation

In 1955 Enrico Fermi, John Pasta, and Stanislaw Ulam carried out a seemingly innocent computer experiment at Los Alamos, [15]. They considered a simple model for a nonlinear one-dimensional crystal describing the motion of a chain of particles with nearest neighbor interaction.



The Hamiltonian of such a system is given by

$$\mathcal{H}(p,q) = \sum_{n\in\mathbb{Z}} \Big(\frac{p(n,t)^2}{2} + V(q(n+1,t) - q(n,t))\Big), \tag{1}$$

where $q(n,t)$ is the displacement of the n-th particle from its equilibrium position, $p(n,t)$ is its momentum (mass $m = 1$), and $V(r)$ is the interaction potential.

Restricting the attention to finitely many particles (e.g., by imposing periodic boundary conditions) and to the **harmonic interaction** $V(r) = \frac{r^2}{2}$, the equations of motion form a linear system of differential equations with constant coefficients. The solution is then given by a superposition of the associated *normal modes*. It was general belief at that time that a generic nonlinear perturbation would yield to *thermalization*. That is, for any initial condition the energy should eventually be equally distributed over all normal modes. The aim of the experiment was to investigate the rate of approach to the equipartition of energy. However, much to everybody's surprise, the experiment indicated, instead of the expected thermalization, a quasi-periodic motion of the system! Many attempts were made to explain this result but it was not until ten years later that Martin Kruskal and Norman Zabusky, [47], revealed the connections with solitons.

This had a big impact on soliton mathematics and led to an explosive growth in the last decades. In particular, it led to the search for a potential $V(r)$ for which the above system has soliton solutions. By considering addition formulas for elliptic functions, Morikazu Toda came up with the choice

$$V(r) = \mathrm{e}^{-r} + r - 1. \tag{2}$$

The corresponding system is now known as the **Toda equation**, [41].

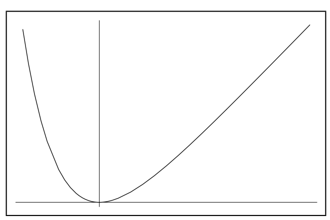


Figure 1: Toda potential $V(r)$

This model is of course only valid as long as the relative displacement is not too large (i.e., at least smaller than the distance of the particles in the equilibrium position). For small displacements it is approximately equal to a harmonic interaction.

The equation of motion in this case reads explicitly

$$
\begin{aligned}
\frac{d}{dt}p(n,t) &= -\frac{\partial\mathcal{H}(p,q)}{\partial q(n,t)} \\
&= \mathrm{e}^{-(q(n,t)-q(n-1,t))} - \mathrm{e}^{-(q(n+1,t)-q(n,t))}, \\
\frac{d}{dt}q(n,t) &= \frac{\partial\mathcal{H}(p,q)}{\partial p(n,t)} = p(n,t).
\end{aligned} \tag{3}
$$

The important property of the Toda equation is the existence of so called **soliton** solutions, that is, pulslike waves which spread in time without changing their size and shape. This is a surprising phenomenon, since for a generic linear equation one would expect spreading of waves (dispersion) and for a generic nonlinear force one would expect that solutions only exist for a finite time (breaking of waves). Obviously our particular force is such that both phenomena cancel each other giving rise to a stable wave existing for all time!

In fact, in the simplest case of one soliton you can easily verify that this solution is given by

$$
q_1(n,t) = q_0 - \ln\frac{1+\gamma\exp(-2\kappa n \pm 2\sinh(\kappa)t)}{1+\gamma\exp(-2\kappa(n-1) \pm 2\sinh(\kappa)t)}, \quad \kappa,\gamma>0. \tag{4}
$$

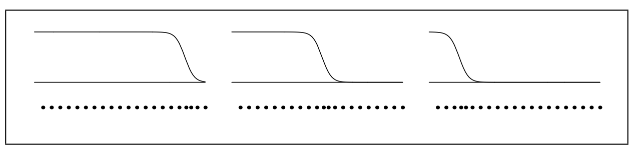


Figure 2: One soliton

It describes a single bump traveling through the crystal with speed $\pm\sinh(\kappa)/\kappa$ and width proportional to $1/\kappa$. In other words, the smaller the soliton the faster it propagates. It results in a total displacement 2κ of the crystal.

Such *solitary waves* were first observed by the naval architect John Scott Russel [34], who followed the bow wave of a barge which moved along a channel maintaining its speed and size (see the review article by Palais [33] for further information).

Existence of soliton solutions is usually connected to complete integrability of the system, and this is also true for the Toda equation.

The motivation as a simple model in solid state physics presented here is of course not the only application of the Toda equation. In fact, the Toda equation and related equations are used to model Langmuir oscillations in plasma physics, to investigate conducting polymers, in quantum cohomology, etc.. Some general books dealing with the Toda lattice are the monographs by Toda [40], [41], by Eilenberger [13], by Faddeev and Takhtajan [14] and by Teschl [39]. Another good source on soliton mathematics is the recent review article by Palais [33]. Finally, it should also be mentioned that the Toda equation can be viewed as a discrete version of the Korteweg-de Vries equation (see [40] or [33] for informal treatments).

2 Complete integrability and Lax pairs

To see that the Toda equation is indeed integrable we introduce Flaschka's variables

$$(5) \qquad a(n,t) = \frac{1}{2}\mathrm{e}^{-(q(n+1,t)-q(n,t))/2}, \qquad b(n,t) = -\frac{1}{2}p(n,t)$$

and obtain the form most convenient for us

$$\begin{aligned} \dot{a}(t) &= a(t)\Big(b^+(t) - b(t)\Big), \\ (6) \qquad \dot{b}(t) &= 2\Big(a(t)^2 - a^-(t)^2\Big). \end{aligned}$$

Here we have used the abbreviation

$$(7) \qquad f^{\pm}(n) = f(n \pm 1).$$

To show complete integrability it suffices to find a so-called **Lax pair**, that is, two operators $H(t)$, $P_2(t)$ in $\ell^2(\mathbb{Z})$ such that the Lax equation

$$(8) \qquad \frac{d}{dt}H(t) = P_2(t)H(t) - H(t)P_2(t)$$

is equivalent to (6). One can easily convince oneself that the right choice is

$$\begin{aligned} H(t) &= a(t)S^+ + a^-(t)S^- + b(t), \\ (9) \qquad P_2(t) &= a(t)S^+ - a^-(t)S^-, \end{aligned}$$

where $(S^{\pm}f)(n) = f^{\pm}(n) = f(n \pm 1)$ are the shift operators. Now the Lax equation (8) implies that the operators $H(t)$ for different $t \in \mathbb{R}$ are unitarily equivalent:

Theorem 1. *Let $P_2(t)$ be a family of bounded skew-adjoint operators, such that $t \mapsto P_2(t)$ is differentiable. Then there exists a family of unitary propagators $U_2(t,s)$ for $P_2(t)$, that is,*

$$(10) \qquad \frac{d}{dt}U_2(t,s) = P_2(t)U_2(t,s), \qquad U_2(s,s) = \mathbb{1}.$$

Moreover, the Lax equation (8) implies

$$(11) \qquad H(t) = U_2(t,s)H(s)U_2(t,s)^{-1}.$$

If the Lax equation (8) holds for $H(t)$ it automatically also holds for $H(t)^j$. Taking traces shows that

$$(12) \qquad \mathrm{tr}\Big(H(t)^j - H_0^j\Big), \quad j \in \mathbb{N},$$

are conserved quantities, where H_0 is the operator corresponding to the constant solution $a_0(n,t) = \frac{1}{2}$, $b_0(n,t) = 0$ (it is needed to make the trace converge). For example,

$$\begin{aligned} \operatorname{tr}\big(H(t) - H_0\big) &= \sum_{n\in\mathbb{Z}} b(n,t) = -\frac{1}{2}\sum_{n\in\mathbb{Z}} p(n,t) \text{ and} \\ \operatorname{tr}\big(H(t)^2 - H_0^2\big) &= \sum_{n\in\mathbb{Z}} b(n,t)^2 + 2(a(n,t)^2 - \frac{1}{4}) = \frac{1}{2}\mathcal{H}(p,q) \qquad (13) \end{aligned}$$

correspond to conservation of the total momentum and the total energy, respectively.

The Lax pair approach was first advocated by Lax [29] in connection with the Korteweg-de Vries equation. The results presented here are due to Flaschka [16], [17]. More informations on the trace formulas and conserved quantities can be found in Gesztesy and Holden [18] and Teschl [37].

3 Types of solutions

The reformulation of the Toda equation as a Lax pair is the key to methods for solving the Toda equation based on spectral and inverse spectral theory for the **Jacobi operator** H (tridiagonal infinite matrix). But before we go into further details let me first show what kind of solutions one can obtain by these methods.

The first type of solution is the general N-soliton solution

$$q_N(n,t) = q_0 - \ln \frac{\det(\mathbb{1} + C_N(n,t))}{\det(\mathbb{1} + C_N(n-1,t))}, \qquad (14)$$

where

$$C_N(n,t) = \left(\frac{\sqrt{\gamma_i\gamma_j}}{1 - \mathrm{e}^{-(\kappa_i+\kappa_j)}} \mathrm{e}^{-(\kappa_i+\kappa_j)n - (\sigma_i \sinh(\kappa_i) + \sigma_j \sinh(\kappa_j))t} \right)_{1\le i,j\le N} \qquad (15)$$

with $\kappa_j, \gamma_j > 0$ and $\sigma_j \in \{\pm 1\}$. The case $N = 1$ coincides with the one soliton solution from the first section. Two examples with $N = 2$ are depicted below. These solutions can be obtained by either factorizing the underlying

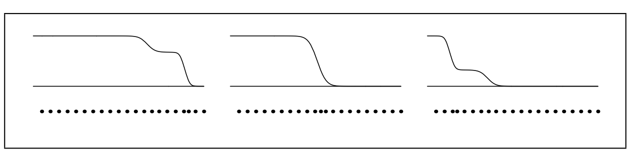


Figure 3: Two solitons, one overtaking the other

Jacobi operator according to $H = AA^*$ and then commuting the factors or, alternatively, by the **inverse scattering transform**.

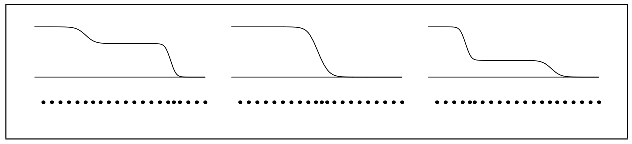


Figure 4: Two solitons traveling in different directions

The second class of solutions are (quasi-)periodic solutions which can be found using techniques from **Riemann surfaces** (respectively algebraic curves). Each such solution is associated with a **hyperelliptic curve** of the type

$$(16) \qquad w^2 = \prod_{j=0}^{2g+1} (z - E_j), \quad E_j \in \mathbb{R},$$

where E_j, $0 \le j \le 2g+1$, are the band edges of the spectrum of H (which is independent of t and hence determined by the initial conditions). One obtains

$$(17) \qquad q(n,t) = q_0 - 2(t\tilde{b} + n\ln(2\tilde{a})) - \ln \frac{\theta(\underline{z}_0 - 2n\underline{A}_{p_0}(\infty_+) - 2t\underline{c}(g))}{\theta(\underline{z}_0 - 2(n-1)\underline{A}_{p_0}(\infty_+) - 2t\underline{c}(g))},$$

where $\underline{z}_0 \in \mathbb{R}^g$, $\theta : \mathbb{R}^g \to \mathbb{R}$ is the Riemann theta function associated with the hyperelliptic curve (16), and $\tilde{a}, \tilde{b} \in \mathbb{R}$, $\underline{A}_{p_0}(\infty_+), \underline{c}(g) \in \mathbb{R}^g$ are constants depending only on the curve (i.e., on E_j, $0 \le j \le 2g+1$). If $q(n,0)$, $p(n,0)$ are (quasi-) periodic with average 0, then $\tilde{a} = \frac{1}{2}$, $\tilde{b} = 0$.

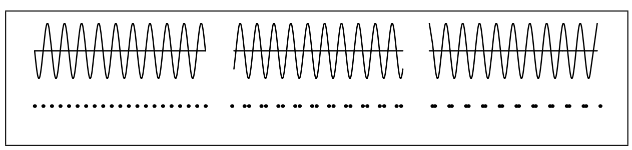


Figure 5: A periodic solution associated with $w^2 = (z^2-2)(z^2-1)$

How these solutions can be obtained will be outlined in the following sections. These methods can also be used to combine both types of solutions and *put N solitons on top* of a given periodic solution.

4 The Toda hierarchy

The Lax approach allows for a straightforward generalization of the Toda equation by replacing P_2 with more general operators P_{2r+2} of order $2r+2$. This yields the **Toda hierarchy**

$$(18) \qquad \frac{d}{dt} H(t) = P_{2r+2}(t)H(t) - H(t)P_{2r+2}(t) \qquad \Leftrightarrow \qquad \mathrm{TL}_r(a,b) = 0.$$

To determine the admissible operators P_{2r+2} (i.e., those for which the commutator with H is of order 2) one restricts them to the algebraic kernel of $H-z$

$$(P_{2r+2}|_{\mathrm{Ker}(H-z)}) = 2aG_r(z)S^+ - H_{r+1}(z), \tag{19}$$

where

$$G_r(z) = \sum_{j=0}^{r} g_{r-j} z^j, \qquad H_{r+1}(z) = z^{r+1} + \sum_{j=0}^{r} h_{r-j} z^j - g_{r+1}. \tag{20}$$

Inserting this into (18) shows after a long and tricky calculation that the coefficients are given by the diagonal and off-diagonal matrix elements of H^j,

$$g_j(n) = \langle \delta_n, H^j \delta_n \rangle, \qquad h_j(n) = 2a(n) \langle \delta_{n+1}, H^j \delta_n \rangle. \tag{21}$$

Here $\langle ., .. \rangle$ denotes the scalar product in $\ell^2(\mathbb{Z})$ and $\delta_n(m) = 1$ for $m = n$ respectively $\delta_n(m) = 0$ for $m \neq n$ is the canonical basis. The r-th Toda equation is then explicitly given by

$$\begin{aligned} \dot{a}(t) &= a(t)(g^+_{r+1}(t) - g_{r+1}(t)), \\ \dot{b}(t) &= h_{r+1}(t) - h^-_{r+1}(t). \end{aligned} \tag{22}$$

The coefficients $g_j(n)$ and $h_j(n)$ can be computed recursively.

The Toda hierarchy was first considered by Ueno and Takasaki [44], [45]. The recursive approach for the standard Lax formalism, [29] was first advocated by Al'ber [2]. Here I followed Bulla, Gesztesy, Holden, and Teschl [8].

5 The Kac-van Moerbeke hierarchy

Consider the **super-symmetric Dirac operator**

$$D(t) = \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \tag{23}$$

and choose

$$A(t) = \rho_o(t) S^+ + \rho_e(t), \qquad A(t)^* = \rho_o^-(t) S^- + \rho_e(t), \tag{24}$$

where

$$\rho_e(n,t) = \rho(2n,t), \qquad \rho_o(n,t) = \rho(2n+1,t) \tag{25}$$

are the "even" and "odd" parts of some bounded sequence $\rho(t)$. Then $D(t)$ is associated with two Jacobi operators

$$H_1(t) = A(t)^* A(t), \qquad H_2(t) = A(t) A(t)^*, \tag{26}$$

whose coefficients read

$$
\begin{aligned}
& a_1(t) = \rho_e(t)\rho_o(t), \qquad && b_1(t) = \rho_e(t)^2 + \rho_o^-(t)^2, \\
(27) \qquad & a_2(t) = \rho_e^+(t)\rho_o(t), \qquad && b_2(t) = \rho_e(t)^2 + \rho_o(t)^2.
\end{aligned}
$$

The corresponding Lax equation

$$
(28) \qquad \frac{d}{dt} D(t) = Q_{2r+2}(t) D(t) - D(t) Q_{2r+2}(t),
$$

where

$$
(29) \qquad Q_{2r+2}(t) = \left(\begin{array}{cc} P_{1,2r+2}(t) & 0 \\ 0 & P_{2,2r+2}(t) \end{array} \right),
$$

gives rise to evolution equations for $\rho(t)$ which are known as the **Kac-van Moerbeke hierarchy**, $\mathrm{KM}_r(\rho) = 0$. The first one (the Kac-van Moerbeke equation) explicitly reads

$$
(30) \qquad \mathrm{KM}_0(\rho) = \dot{\rho}(t) - \rho(t)(\rho^+(t)^2 - \rho^-(t)^2) = 0.
$$

Moreover, from the way we introduced the Kac-van Moerbeke hierarchy, it is not surprising that there is a close connection with the Toda hierarchy. To reveal this connection all one has to do is to insert

$$
(31) \qquad D(t)^2 = \left(\begin{array}{cc} H_1(t) & 0 \\ 0 & H_2(t) \end{array} \right)
$$

into the Lax equation

$$
(32) \qquad \frac{d}{dt} D(t)^2 = Q_{2r+2}(t) D(t)^2 - D(t)^2 Q_{2r+2}(t),
$$

which shows that the Lax equation (28) for $D(t)$ implies the Lax equation (18) for both H_1 and H_2. This observation gives a **Bäcklund transformation** between the Kac-van Moerbeke and the Toda hierarchies:

Theorem 2. *For any given solution $\rho(t)$ of $\mathrm{KM}_r(\rho) = 0$ we obtain, via (27), two solutions $(a_j(t), b_j(t))_{j=1,2}$ of $\mathrm{TL}_r(a,b) = 0$.*

This is the analog of the Miura transformation between the modified and the original Korteweg-de Vries hierarchies.

The Kac-van Moerbeke equation has been first introduced by Kac and van Moerbeke in [23]. The Bäcklund transformation connecting the Toda and the Kac-van Moerbeke equations has first been considered by Toda and Wadati in [43].

6 Commutation methods

Clearly, it is natural to ask whether this transformation can be inverted. In other words, can we factor a given Jacobi operator H as A^*A and then compute the corresponding solution of the Kac-van Moerbeke hierarchy plus the second solution of the Toda hierarchy?

This can in fact be done. All one needs is a positive solution of the system

$$H(t)u(n,t) = 0, \qquad \frac{d}{dt}u(n,t) = P_{2r+2}(t)u(n,t) \tag{33}$$

and then one has

$$\begin{aligned} \rho_o(t) &= -\sqrt{\frac{-a(t)u(t)}{u^+(t)}}, \\ \rho_e(t) &= \sqrt{\frac{-a(t)u^+(t)}{u(t)}}. \end{aligned} \tag{34}$$

In particular, starting with the trivial solution $a_0(n,t) = -\frac{1}{2}$, $b_0(n,t) = 0$ and proceeding inductively one ends up with the N-soliton solutions.

The method of factorizing H and then commuting the factors is known as **Darboux transformation** and is of independent interest since it has the property of inserting a single eigenvalue into the spectrum of H.

Commutation methods for Jacobi operators in connection with the Toda and Kac-van Moerbeke equation were first considered by Gesztesy, Holden, Simon, and Zhao [22]. For further generalizations, see Gesztesy and Teschl [20] and Teschl [38]. A second way to obtain the N-soliton solution is via the inverse scattering transform, which was first worked out by Flaschka in [17].

7 Stationary solutions

In the remaining sections I would like to show how two at first sight unrelated fields of mathematics, spectral theory and algebraic geometry, can be combined to find (quasi-)periodic solutions of the Toda equations.

To reveal this connection, we first look at stationary solutions of the Toda hierarchy or, equivalently, at commuting operators

$$P_{2r+2}H - HP_{2r+2} = 0. \tag{35}$$

In this case a short calculation gives

$$(P_{2r+2}|_{\mathrm{Ker}(H-z)})^2 = H_{r+1}(z)^2 - 4a^2 G_r(z)G_r^+(z) =: R_{2r+2}(z), \tag{36}$$

where $R_{2r+2}(z)$ can be shown to be independent of n. That is, it is of the form

$$R_{2r+2}(z) = \prod_{j=0}^{2r+1}(z - E_j) \tag{37}$$

for some constant numbers $E_j \in \mathbb{R}$. In particular, this implies

$$(P_{2r+2})^2 = \prod_{j=0}^{2r+1} (H - E_j) \tag{38}$$

and the polynomial $w^2 = \prod_{j=0}^{2r+1}(z - E_j)$ is known as the **Burchnall-Chaundy polynomial** of P_{2r+2} and H. In particular, the connection between the stationary Toda hierarchy and the **hyperelliptic curve**

$$\mathcal{K} = \{(z, w) \in \mathbb{C}^2 | w^2 = \prod_{j=0}^{2r+1} (z - E_j)\} \tag{39}$$

is apparent. But how can it be used to solve the Toda equation? This will be shown next. We will for simplicity assume that our curve is nonsingular, that is, that $E_j < E_{j+1}$ for all j.

The fact that two commuting differential or difference operators satisfy a polynomial relation, was first shown by Burchnall and Chaundy [9], [10]. The approach to stationary solutions presented here follows again Bulla, Gesztesy, Holden, and Teschl [8].

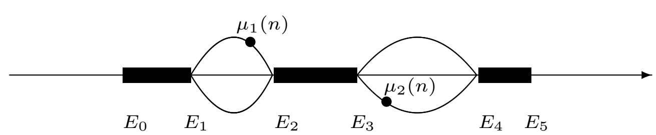
8 Jacobi operators associated with stationary solutions

Next some spectral properties of the Jacobi operators associated with stationary solutions are needed. First of all, one can show that

$$\begin{aligned} g(z,n) &= \frac{G_r(z,n)}{R_{2r+2}^{1/2}(z)} = \langle \delta_n, (H-z)^{-1}\delta_n \rangle, \\ h(z,n) &= \frac{H_{r+1}(z,n)}{R_{2r+2}^{1/2}(z)} = \langle \delta_{n+1}, (H-z)^{-1}\delta_n \rangle. \end{aligned} \tag{40}$$

This is not too surprising, since g_j and h_j are by (21) just the expansion coefficients in the Neumann series of the resolvent.

But once we know the diagonal of the resolvent we can easily read off the spectrum of H. The open branch cuts of $R_{2r+2}^{1/2}(z)$ form an essential support of the absolutely continuous spectrum and the branch points support the singular spectrum. Since at each branch point we have a square root singularity, there can be no eigenvalues and since the singular continuous spectrum cannot be supported on finitely many points, the spectrum is purely absolutely continuous and consists of a finite number of bands.



The points $\mu_j(n)$ are the zeros of $G_r(z,n)$,

$$G_r(z,n) = \prod_{j=1}^{r} (z - \mu_j(n)), \tag{41}$$

and can be interpreted as the eigenvalues of the operator H_n obtained from H by imposing an additional **Dirichlet boundary condition** $u(n) = 0$ at n. Since H_n decomposes into a direct sum $H_{-,n} \oplus H_{+,n}$ we can also associate a sign $\sigma_j(n)$ with $\mu_j(n)$, indicating whether it is an eigenvalue of $H_{-,n}$ or $H_{+,n}$.

Theorem 3. *The band edges $\{E_j\}_{0\le j\le 2r+1}$ together with the Dirichlet data $\{(\mu_j(n),\ \sigma_j(n))\}_{1\le j\le r}$ for one n uniquely determine H. Moreover, it is even possible to write down explicit formulas for $a(n+k)$ and $b(n+k)$ for all $k \in \mathbb{Z}$ as functions of these data. Explicitly one has*

$$\begin{aligned}
b(n) &= b^{(1)}(n)\\
a(n-{\textstyle{0 \atop 1}})^2 &= \frac{b^{(2)}(n) - b(n)^2}{4} \pm \sum_{j=1}^{r} \frac{\sigma_j(n) R_{2r+2}^{1/2}(\mu_j(n))}{2\prod_{k\neq j}(\mu_j(n)-\mu_k(n))}\\
b(n\pm 1) &= \frac{1}{a(n-{\textstyle{0 \atop 1}})^2}\Big(\frac{2b^{(3)}(n) - 3b(n)b^{(2)}(n) + b(n)^3}{12}\\
&\qquad \pm \sum_{j=1}^{r} \frac{\sigma_j(n) R_{2r+2}^{1/2}(\mu_j(n))\mu_j(n)}{2\prod_{k\neq j}(\mu_j(n)-\mu_k(n))}\Big)\\
&\quad \vdots
\end{aligned} \tag{42}$$

where

$$b^{(\ell)}(n) = \frac{1}{2}\sum_{j=0}^{2r+1} E_j^{\ell} - \sum_{j=1}^{r} \mu_j(n)^{\ell}. \tag{43}$$

These formulas already indicate that $\hat{\mu}_j(n) = (\mu_j(n), \sigma_j(n))$ should be considered as a point on the Riemann surface $\mathcal{K}$ of $R_{2r+2}^{1/2}(z)$, where $\sigma_j(n)$ indicates on which sheet $\mu_j(n)$ lies.

The result for periodic operators is due to van Moerbeke [30], the general case was given by Gesztesy, Krishna, and Teschl [21]. Trace formulas for Sturm-Liouville and also for Jacobi operators have a long history. The formulas for $b^{(\ell)}$, $\ell = 1, 2$, were already given in [30] for the periodic case. The formulas presented here and in particular the fact that the coefficients a and b can be explicitly written down in terms of minimal spectral data are due to Teschl [36]. Most proofs use results on orthogonal polynomials and the moment problem. One of the classical references is [1], for a recent review article see Simon [35].

9 Algebro-geometric solutions of the Toda equations

The idea now is to choose a stationary solution of $\mathrm{TL}_r(a,b)=0$ as the initial condition for $\mathrm{TL}_s(a,b)$ and to consider the time evolution in our new *coordinates* $\{E_j\}_{0\le j\le 2r+1}$ and $\{(\mu_j(n),\sigma_j(n))\}_{1\le j\le r}$. From unitary equivalence of the family of operators $H(t)$ we know that the band edges E_j do not depend on t. Moreover, the time evolution of the Dirichlet data follows from the Lax equation

$$(44)\qquad \frac{d}{dt}(H(t)-z)^{-1} = [P_{2s+2}(t),(H(t)-z)^{-1}].$$

Inserting (40) and (41) yields

$$(45)\qquad \frac{d}{dt}\mu_j(n,t) = -2G_s(\mu_j(n,t),n,t)\frac{\sigma_j(n,t)R_{2r+2}^{1/2}(\mu_j(n,t))}{\prod_{k\neq j}(\mu_k(n,t)-\mu_j(n,t))},$$

where $G_s(z)$ has to be expressed in terms of μ_j using (42). Again, this equation should be viewed as a differential equation on $\mathcal{K}$ rather than $\mathbb{R}$. A closer investigation shows that each Dirichlet eigenvalue $\mu_j(n,t)$ rotates in its spectral gap.

At first sight it looks like we have not gained too much since this flow is still highly nonlinear, but it can be straightened out using Abel's map from algebraic geometry. So let us review some basic facts first.

Our hyperelliptic curve $\mathcal{K}$ is in particular a compact Riemann surface of genus r and hence it has a basis of r holomorphic differentials which are explicitly given by

$$(46)\qquad \zeta_j = \sum_{k=1}^{r} c_j(k)\frac{z^{k-1}dz}{R_{2r+2}^{1/2}(z)}.$$

(At first sight these differentials seem to have poles at each band edge, but near such a band edge we need to use a chart $z-E_j=w^2$ and $dz=2wdw$ shows that each zero in the denominator cancels with a zero in the numerator). Given a homology basis a_j, b_j for $\mathcal{K}$ they are usually normalized such that

$$(47)\qquad \int_{a_j}\zeta_k = \delta_{j,k} \qquad \text{and one sets} \qquad \int_{b_j}\zeta_k =: \tau_{jk}.$$

Now the **Jacobi variety** associated with $\mathcal{K}$ is the r-dimensional torus $\mathbb{C}^r \bmod L$, where $L=\mathbb{Z}^r+\tau\mathbb{Z}^r$ and the **Abel map** is given by

$$(48)\qquad \underline{A}_{p_0}(p) = \int_{p_0}^{p}\underline{\zeta} \bmod L, \qquad p,p_0\in\mathcal{K}.$$

Theorem 4. *The Abel map straightens out the dynamical system $\hat{\mu}_j(0,0) \to \hat{\mu}_j(n,t)$ both with respect to n and t*

$$\sum_{j=1}^{r} \underline{A}_{p_0}(\hat{\mu}_j(n,t)) = \sum_{j=1}^{r} \underline{A}_{p_0}(\hat{\mu}_j(0,0)) - 2n\underline{A}_{p_0}(\infty_+) - t\underline{U}_s, \tag{49}$$

where $\underline{U}_s$ can be computed explicitly in terms of the band edges E_j.

Sketch of proof. Consider the function (compare (19))

$$\phi(p,n,t) = \frac{H_{r+1}(p,n,t) + R_{2r+2}^{1/2}(p)}{2a(n,t)G_r(p,n,t)} = \frac{2a(n,t)G_r(p,n+1,t)}{H_{r+1}(p,n,t) - R_{2r+2}^{1/2}(p)}, \quad p \in \mathcal{K}, \tag{50}$$

whose zeros are $\hat{\mu}_j(n+1,t)$, ∞_- and whose poles are $\hat{\mu}_j(n,t)$, ∞_+. Abel's theorem implies

$$\underline{A}_{p_0}(\infty_+) + \sum_{j=1}^{r} \underline{A}_{p_0}(\hat{\mu}_j(n,t)) = \underline{A}_{p_0}(\infty_-) + \sum_{j=1}^{r} \underline{A}_{p_0}(\hat{\mu}_j(n+1,t)), \tag{51}$$

which settles the first claim. To show the second claim we compute

$$\begin{aligned} \frac{d}{dt}\sum_{j=1}^{r} \underline{A}_{p_0}(\hat{\mu}_j) &= \sum_{j=1}^{r} \dot{\mu}_j \sum_{k=1}^{r} \underline{c}(k) \frac{\mu_j^{k-1}}{\sigma_j R_{2r+2}^{1/2}(\mu_j)} \\ &= -2\sum_{j,k}^{r} \underline{c}(k) \frac{G_s(\mu_j)}{\prod_{\ell\neq j}(\mu_j - \mu_\ell)} \mu_j^{k-1}. \end{aligned} \tag{52}$$

The key idea is now to rewrite this as an integral

$$\int_\Gamma \frac{G_s(z)}{G_r(z)} z^{k-1} dz, \tag{53}$$

where Γ is a closed path encircling all points μ_j. By (41) this is equal to the above expression by the residue theorem. Moreover, since the integrand is rational we can also compute this integral by evaluating the residue at ∞, which is given by

$$\begin{aligned} \frac{G_s(z)}{G_r(z)} &= \frac{G_s(z)}{g(z)} \frac{1}{R_{2r+2}^{1/2}(z)} = \frac{z^{s+1}(1 + O(z^{-s}))}{R_{2r+2}^{1/2}(z)} \\ &= -2 \sum_{\ell=\max\{1,r-s\}} \underline{c}(\ell) d_{s-r+\ell}(\underline{E}) =: \underline{U}_s, \end{aligned} \tag{54}$$

since the coefficients of G_s coincide with the first s coefficients in the Neumann series of $g(z)$ by (21). Here $d_j(\underline{E})$ are just the coefficients in the asymptotic expansion of $1/R_{2r+2}^{1/2}(z)$. □

Since the poles and zeros of the function $\phi(z)$, which appeared in the proof of the last theorem, as well as their image under the Abel map are known, a function having the same zeros and poles can be written down using **Riemann theta functions** (**Jacobi's inversion problem** and **Riemann's vanishing theorem**). The **Riemann–Roch theorem** implies that both functions coincide. Finally, the function $\phi(z)$ has also a spectral interpretation as Weyl m-function, and thus explicit formulas for the coefficients a and b can be obtained from the asymptotic expansion for $|z| \to \infty$. This produces the formula in equation (17).

The first results on algebro-geometric solutions of the Toda equation were given by Date and Tanaka [11]. Further important contributions were made by Krichever, [24] – [28], van Moerbeke and Mumford [31], [32]. The presentation here follows Bulla, Gesztesy, Holden, and Teschl [8] respectively Teschl [39]. Another possible approach is to directly use the spectral function of H and to consider its t dependence, see Berezanskiĭ and coworkers [3]–[7]. For some recent developments based on Lie theoretic methods and loop groups I again recommend the review by Palais [33] as starting point.

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References

[1] N. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, London, 1965.

[2] S. J. Al'ber, *Associated integrable systems*, J. Math. Phys. **32**, 916–922 (1991).

[3] Yu. M. Berezanskiĭ, *Integration of nonlinear difference equations by the inverse spectral problem method*, Soviet Math. Dokl., **31** No. 2, 264–267 (1985).

[4] Yu. M. Berezanski, *The integration of semi-infinite Toda chain by means of inverse spectral problem*, Rep. Math. Phys., **24** No. 1, 21–47 (1985).

[5] Yu. M. Berezansky, *Integration of nonlinear nonisospectral difference-differential equations by means of the inverse spectral problem*, in "*Nonlinear Physics. Theory and experiment*", (eds E. Alfinito, M. Boiti, L. Martina, F. Pempinelli), World Scientific, 11–20 (1996).

[6] Yu. M. Berezansky and M. I. Gekhtman, *Inverse problem of the spectral analysis and non-Abelian chains of nonlinear equations*, Ukrain. Math. J., **42**, 645–658 (1990).

[7] Yu. Berezansky and M. Shmoish, *Nonisospectral flows on semi-infinite Jacobi matrices*, Nonl. Math. Phys., **1** No. 2, 116–146 (1994).

[8] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, Mem. Amer. Math. Soc. **135-641**, 1998.

[9] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. Ser. 2, **21**, 420–440 (1923).

[10] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. Roy. Soc. London **A118**, 557–583 (1928).

[11] E. Date and S. Tanaka *Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice*, Prog. Theoret. Phys. **56**, 457–465 (1976).

[12] P. Deift, L.C. Li, and C. Tomei, *Toda flows with infinitely many variables*, J. Func. Anal. **64**, 358–402 (1985).

[13] S. N. Eilenberger, *An Introduction to Difference Equations*, Springer, New York, 1996.

[14] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[15] E. Fermi, J. Pasta, S. Ulam, *Studies of Nonlinear Problems*, Collected Works of Enrico Fermi, University of Chicago Press, Vol.II,978–988,1965. Theory, Methods, and Applications, 2nd ed., Marcel Dekker, New York, 2000.

[16] H. Flaschka, *The Toda lattice. I. Existence of integrals*, Phys. Rev. B **9**, 1924–1925 (1974).

[17] H. Flaschka, *On the Toda lattice. II. Inverse-scattering solution*, Progr. Theoret. Phys. **51**, 703–716 (1974).

[18] F. Gesztesy and H. Holden, *Trace formulas and conservation laws for nonlinear evolution equations*, Rev. Math. Phys. **6**, 51–95 (1994).

[19] F. Gesztesy and H. Holden, *Soliton Equations and their Algebro-Geometric Solutions I – III*, Cambridge Series in Advanced Mathematics, in preparation.

[20] F. Gesztesy and G. Teschl, *Commutation methods for Jacobi operators*, J. Diff. Eqs. **128**, 252–299 (1996).

[21] F. Gesztesy, M. Krishna, and G. Teschl, *On isospectral sets of Jacobi operators*, Com. Math. Phys. **181**, 631–645 (1996).

[22] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, *On the Toda and Kac-van Moerbeke systems*, Trans. Amer. Math. Soc. **339**, 849–868 (1993).

[23] M. Kac and P. van Moerbeke, *On an explicitly soluble system of nonlinear differential equations, related to certain Toda lattices*, Adv. Math. **16**, 160–169 (1975).

[24] I. M. Krichever, *Algebraic curves and nonlinear difference equations*, Russian Math. Surveys. **334**, 255–256 (1978).

[25] I. M. Krichever, *Nonlinear equations and elliptic curves*, Rev. of Science and Technology **23**, 51–90 (1983).

[26] I. M. Krichever, *Algebro-geometric spectral theory of the Schrödinger difference operator and the Peierls model*, Soviet Math. Dokl. **26**, 194–198 (1982).

[27] I. M. Krichever, *The Peierls model*, Funct. Anal. Appl. **16**, 248–263 (1982).

[28] I. Krichever, *Algebraic-geometrical methods in the theory of integrable equations and their perturbations*, Acta Appl. Math. **39**, 93–125 (1995).

[29] P. D. Lax *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math. **21**, 467–490 (1968).

[30] P. van Moerbeke, *The spectrum of Jacobi Matrices*, Inv. Math. **37**, 45–81 (1976).

[31] P. van Moerbeke and D. Mumford *The spectrum of difference operators and algebraic curves*, Acta Math. **143**, 97–154 (1979).

[32] D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations*, Intl. Symp. Algebraic Geometry, 115–153, Kyoto, 1977.

[33] R. S. Palais, *The symmetries of solitons*, Bull. Amer. Math. Soc., **34**, 339–403 (1997).

[34] J. S. Russel, *Report on waves*, 14th Mtg. of the British Assoc. for the Advance of Science, John Murray, London, pp. 311–390 + 57 plates, 1844.

[35] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Advances in Math. **137**, 82–203 (1998).

[36] G. Teschl, *Trace Formulas and Inverse Spectral Theory for Jacobi Operators*, Comm. Math. Phys. **196**, 175–202 (1998).

[37] G. Teschl, *Inverse scattering transform for the Toda hierarchy*, Math. Nach. **202**, 163–171 (1999).

[38] G. Teschl, *On the Toda and Kac-van Moerbeke hierarchies*, Math. Z. **231**, 325–344 (1999).

[39] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Monographs **72**, Amer. Math. Soc., Rhode Island, 2000

[40] M. Toda, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.

[41] M. Toda, *Theory of Nonlinear Waves and Solitons*, Kluwer, Dordrecht, 1989.

[42] M. Toda, *Selected Papers of Morikazu Toda*, ed. by M. Wadati, World Scientific, Singapore, 1993.

[43] M. Toda and M. Wadati, *A canonical transformation for the exponential lattice*, J. Phys. Soc. Jpn. **39**, 1204–1211 (1975).

[44] K. Ueno and K. Takasaki, *Toda lattice hierarchy*, in "*Advanced Studies in Pure Mathematics 4*", (ed. K. Okamoto), North-Holland, Amsterdam, 1–95 (1984).

[45] K. Ueno and K. Takasaki, *Toda lattice hierarchy. I*', Proc. Japan Acad., Ser. A **59**, 167–170 (1983).

[46] K. Ueno and K. Takasaki, *Toda lattice hierarchy. II*', Proc. Japan Acad., Ser. A **59**, 215–218 (1983).

[47] N. J. Zabusky and M. D. Kruskal, *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett. **15**, 240–243 (1965).

Gerald Teschl
Institut für Mathematik
Strudlhofgasse 4
A-1090 Wien
Email: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/˜gerald/
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