PERTURBATIONS OF PERIODIC STURM–LIOUVILLE OPERATORS

JUSSI BEHRNDT, PHILIPP SCHMITZ, GERALD TESCHL, AND CARSTEN TRUNK

Abstract. We study perturbations of the self-adjoint periodic Sturm–Liouville operator

$$A_0 = \frac{1}{r_0}\left(-\frac{\mathrm{d}}{\mathrm{d}x} p_0 \frac{\mathrm{d}}{\mathrm{d}x} + q_0\right)$$

and conclude under L^1-assumptions on the differences of the coefficients that the essential spectrum and absolutely continuous spectrum remain the same. If a finite first moment condition holds for the differences of the coefficients, then at most finitely many eigenvalues appear in the spectral gaps. This observation extends a seminal result by Rofe-Beketov from the 1960s. Finally, imposing a second moment condition we show that the band edges are no eigenvalues of the perturbed operator.

1. Introduction

Consider a periodic Sturm–Liouville differential expression of the form

$$\tau_0 = \frac{1}{r_0}\left(-\frac{\mathrm{d}}{\mathrm{d}x} p_0 \frac{\mathrm{d}}{\mathrm{d}x} + q_0\right)$$

on $\mathbb{R}$, where $1/p_0, q_0, r_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ are real-valued and ω-periodic, and $r_0 > 0$, $p_0 > 0$ a. e. Let A_0 be the corresponding self-adjoint operator in the weighted L^2-Hilbert space $L^2(\mathbb{R}; r_0)$ and recall that the spectrum of A_0 is semibounded from below, purely absolutely continuous and consists of (finitely or infinitely many) spectral bands; cf. [1], [9] or [15, Section 12].

Now let $1/p_1, q_1, r_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ be real-valued with $r_1 > 0$, $p_1 > 0$ a. e., assume that the condition

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t|^k \,\mathrm{d}t < \infty \tag{1.1}$$

holds for some $k \geq 0$, and consider the corresponding perturbed Sturm–Liouville differential expression

$$\tau_1 = \frac{1}{r_1}\left(-\frac{\mathrm{d}}{\mathrm{d}x} p_1 \frac{\mathrm{d}}{\mathrm{d}x} + q_1\right)$$

on $\mathbb{R}$. It turns out that τ_1 is in the limit point case at both singular endpoints $\pm\infty$ and hence there is a unique self-adjoint realization A_1 of τ_1 in the weighted L^2-Hilbert space $L^2(\mathbb{R}; r_1)$. The first observation in Theorem 1.1 below is that the essential spectra of A_0 and A_1 coincide and the interior is purely absolutely

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2020 *Mathematics Subject Classification.* Primary 34L05, 81Q10; Secondary 34L40, 47E05.

Key words and phrases. Periodic Sturm–Liouville operators, perturbations, essential spectrum, absolutely continuous spectrum, spectral gaps, discrete eigenvalues.

Adv. Math. **422**, 109022, 22 pp (2023).

continuous spectrum of A_1. In the special case $r_0 = r_1 = p_0 = p_1 = 1$ this result is known from [13] and for $p_0 \neq p_1$ a related result is contained in [1]; cf. Remark 2.2.

Theorem 1.1. *Assume that condition* (1.1) *holds for* $k = 0$ *and let* A_0 *and* A_1 *be the self-adjoint realizations of* τ_0 *and* τ_1 *in* $L^2(\mathbb{R}; r_0)$ *and* $L^2(\mathbb{R}; r_1)$*, respectively. Then we have*

$$\sigma_{\mathrm{ess}}(A_0) = \sigma_{\mathrm{ess}}(A_1),$$

the spectrum of A_1 *is purely absolutely continuous in the interior of the spectral bands, and* A_1 *is semibounded from below.*

In particular, the band structure of the spectrum of the periodic operator A_0 is preserved for the essential spectrum of A_1 and in the gaps of $\sigma_{\mathrm{ess}}(A_1)$ discrete eigenvalues may appear that may accumulate to the edges of the spectral bands; for a detailed discussion in the case $r_0 = r_1$ we refer to [1, Section 5.3]. Our second main objective in this note is to verify that under a finite first moment condition on the difference of the coefficients there are at most finitely many discrete eigenvalues in the gaps of the essential spectrum of A_1. The question whether eigenvalues accumulate at the band edges has a long tradition going back to the seminal results of Rofe-Beketov [10], which were later extended by Schmidt [11] (see also [1, §5.4] for the special case $r_0 = r_1 = 1$ and $p_0 = p_1$). They play also an important role for the scattering theory in this setting [2, 3, 4, 7]. The currently best results in this direction can be found in [8], which apply in the special case $r_0 = r_1$.

Theorem 1.2. *Assume that condition* (1.1) *holds for* $k = 1$ *and let* A_0 *and* A_1 *be the self-adjoint realizations of* τ_0 *and* τ_1 *in* $L^2(\mathbb{R}; r_0)$ *and* $L^2(\mathbb{R}; r_1)$*, respectively. Then every gap of the spectral bands* $\sigma_{\mathrm{ess}}(A_0) = \sigma_{\mathrm{ess}}(A_1)$ *contains at most finitely many eigenvalues of* A_1*.*

In the third result we pay special attention to the edges of the spectral bands. If (1.1) holds for $k = 1$ (and hence also for $k = 0$), then the interior of the spectral bands consists of purely absolutely continuous spectrum of A_1 and the eigenvalues of A_1 in the gaps do not accumulate to the band edges. If we further strengthen the assumptions and impose a finite second moment condition $k = 2$ in (1.1) (and hence also $k = 1$ and $k = 0$), then it turns out that the band edges are no eigenvalues of A_1.

Theorem 1.3. *Assume that condition* (1.1) *holds for* $k = 2$ *and let* A_0 *and* A_1 *be the self-adjoint realizations of* τ_0 *and* τ_1 *in* $L^2(\mathbb{R}; r_0)$ *and* $L^2(\mathbb{R}; r_1)$*, respectively. Then the edges of the spectral bands* $\sigma_{\mathrm{ess}}(A_0) = \sigma_{\mathrm{ess}}(A_1)$ *are no eigenvalues of* A_1 *and the spectral bands consist of purely absolutely continuous spectrum of* A_1*.*

In Section 2 we also show that the claim in Theorem 1.3 remains valid if (1.1) holds for $k = 1$ and some other additional assumptions for r_1 and q_1 are satisfied; cf. Proposition 2.5. Our proofs of Theorems 1.1–1.3 are based on a careful analysis of the solutions of $(\tau_0 - \lambda)u = 0$ and $(\tau_1 - \lambda)u = 0$ for $\lambda \in \mathbb{R}$; cf. Lemma 2.6 and Lemma 2.7. While the properties of the solutions of the periodic problem in Lemma 2.6 are mainly consequences of well-known properties of the Hill discriminant, the properties of the solutions of the perturbed problem in Lemma 2.7 require some slightly more technical arguments. It is convenient to first verify variants of Theorems 1.1–1.3 for self-adjoint realizations of τ_0 and τ_1 on half-lines $(-\infty, a)$ and

(a,∞) with finite endpoint a, and use a coupling argument to conclude the corresponding results on $\mathbb{R}$. One of the key ingredients is the connection of the zeros of a modified Wronskian with the finiteness of the spectrum from [5].

Acknowledgments. We are indebted to Fritz Gesztesy for very helpful discussions and literature hints.

2. Perturbations of periodic Sturm–Liouville operators on a half-line

We prove variants of Theorems 1.1–1.3 for self-adjoint realizations H_0 and H_1 of τ_0 and τ_1, respectively, in the L^2-spaces $L^2((a,\infty);r_0)$ and $L^2((a,\infty);r_1)$ with some finite endpoint a. For the real-valued coefficients we have $1/p_j, q_j, r_j \in L^1_{\mathrm{loc}}([a,\infty))$ and $r_j > 0$, $p_j > 0$ a. e., and $1/p_0, q_0, r_0$ are ω-periodic.

The differential expression τ_0 is in the limit point case at ∞ and regular at a. In the following let H_0 be any self-adjoint realization of τ_0 in $L^2((a,\infty);r_0)$. Similar as in the full line case also on the half-line the essential spectrum of H_0 is purely absolutely continuous and consists of infinitely many closed intervals

$$\sigma_{\mathrm{ess}}(H_0) = \bigcup_{k=1}^{\infty} [\lambda_{2k-1}, \lambda_{2k}], \tag{2.1}$$

where the endpoints λ_{2k-1} and λ_{2k}, $\lambda_{2k-1} < \lambda_{2k}$, denote the k-th eigenvalues of the regular Sturm–Liouville operator in $L^2((a,a+\omega);r_0)$ (in nondecreasing order) with periodic and semiperiodic boundary conditions, respectively; cf. [1] or [15, Section 12] for more details. Recall that the closed intervals may adjoin and that also $\sigma_{\mathrm{ess}}(H_0) = [\lambda_1, \infty)$ may happen in (2.1). Each interval $(-\infty, \lambda_1)$ and $(\lambda_{2k}, \lambda_{2k+1})$, $k \in \mathbb{N}$, may contain at most one (simple) eigenvalue of H_0. In particular, H_0 is semibounded from below and (2.1) implies that the interior of $\sigma_{\mathrm{ess}}(H_0)$ is nonempty.

Theorem 2.1. *Assume that*

$$\int_a^{\infty} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) \mathrm{d}t < \infty \tag{2.2}$$

and let H_0 and H_1 be arbitrary self-adjoint realizations of τ_0 and τ_1 in $L^2((a,\infty);r_0)$ and $L^2((a,\infty);r_1)$, respectively. Then we have

$$\sigma_{\mathrm{ess}}(H_0) = \sigma_{\mathrm{ess}}(H_1),$$

the spectrum of H_1 is purely absolutely continuous in the interior of the spectral bands, and H_1 is semibounded from below.

It follows that H_1 has non-empty essential spectrum, hence, the differential expression τ_1 is in the limit point case at ∞.

Remark 2.2. For the special case $r_0 = r_1 = p_0 = p_1 = 1$ the result in Theorem 2.1 goes back to the paper [13] of G. Stolz, where instead of the assumption $q_1 - q_0 \in L^1(a,\infty)$ in (2.2) the weaker conditions

$$\int_c^{\infty} |(q_1 - q_0)(t+\omega) - (q_1 - q_0)(t)| \, \mathrm{d}t < \infty \tag{2.3}$$

for some $c > a$ and

$$\lim_{x\to\infty} \int_x^{x+1} |q_1(t) - q_0(t)| \, \mathrm{d}t = 0 \tag{2.4}$$

are imposed. The considerations from [13] are extended in [1, Chapter 5.2] to the case $r_0 = r_1$ and $p_0 \neq p_1$ with $1/p_1 - 1/p_0$ satisfying similar conditions (2.3)–(2.4). More precisely, in [1, Corollary 5.2.3] it was shown that the interior of the essential spectrum of H_0 is purely absolutely continuous spectrum of H_1 and hence $\sigma_{\text{ess}}(H_0) \subset \sigma_{\text{ess}}(H_1)$. For the other inclusion in [1, Theorem 5.3.1] it is assumed that $r_0 = r_1$, $p_0 = p_1$ together with additional limit conditions for $q_1 - q_0$. For details we refer to [1, Chapter 5].

In the next theorem we strengthen the assumptions by imposing a finite first moment condition (see (2.5) below) on the differences of the coefficients; note that (2.5) implies (2.2) since the coefficients (and their differences) are integrable at a. In this situation it turns out that there appear at most finitely many simple eigenvalues of H_1 in each spectral gap and hence there is no accumulation of eigenvalues to the edges of the band gaps. Concerning the history of this result we refer to the discussion before the corresponding result on $\mathbb{R}$, Theorem 1.2.

Theorem 2.3. *Assume that*

$$\int_a^\infty \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t| \, \mathrm{d}t < \infty \tag{2.5}$$

holds, and let H_1 be an arbitrary self-adjoint realization of τ_1 in $L^2((a,\infty); r_1)$. Then every gap of $\sigma_{\text{ess}}(H_1)$ contains at most finitely many eigenvalues.

In the next result we assume a stronger integrability condition and conclude that the edges of the spectral bands are no embedded eigenvalues of A_1; note that (2.6) implies (2.5) and (2.2). As pointed out before, this question is important for scattering theory and was first established by Firsova [2, 3] in the case $r_0 = r_1 = p_0 = p_1 = 1$.

Theorem 2.4. *Assume that*

$$\int_a^\infty \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t|^2 \, \mathrm{d}t < \infty \tag{2.6}$$

holds, and let H_1 be an arbitrary self-adjoint realization of τ_1 in $L^2((a,\infty); r_1)$. Then the edges of the spectral bands are no eigenvalues of H_1 and the spectral bands consist of purely absolutely continuous spectrum of H_1.

We find it worthwhile to provide another set of assumptions that also imply absence of eigenvalues at the edges of the spectral bands. Here we only assume the integrability condition (2.5), but for r_1 and q_1 additional assumptions are required. It is left to the reader to formulate a variant of Proposition 2.5 for the self-adjoint realization A_1 of τ_1 in $L^2(\mathbb{R}; r_1)$.

Proposition 2.5. *Assume that* (2.5) *holds and that there exist positive constants C_0, C_1 such that r_1 and q_1 satisfy $C_0 \le r_1(t) \le C_1$ and $\int_{t-1}^{t+1} |q_1(s)|^2 ds \le C_1$ for t in some neighbourhood of ∞. Let H_1 be an arbitrary self-adjoint realization of τ_1 in $L^2((a,\infty); r_1)$. Then the edges of the spectral bands are no eigenvalues of H_1 and the spectral bands consist of purely absolutely continuous spectrum of H_1.*

The proofs of Theorem 2.1, Theorem 2.3, Theorem 2.4, and Proposition 2.5 are at the end of this section. In what follows, we investigate solutions of the periodic and the perturbed periodic problem. The first lemma is more or less a variant of standard working knowledge in periodic differential operators and is essentially contained in [1, Chapter 1] or [15]. For the convenience of the reader we provide a short proof.

Lemma 2.6. *For $\lambda \in \mathbb{R}$ there exist linearly independent solutions $u_0 = u_0(\cdot, \lambda)$ and $v_0 = v_0(\cdot, \lambda)$ of $(\tau_0 - \lambda)u = 0$ and $c = c(\lambda) \in \mathbb{C}$ such that the functions $U_0 = U_0(\cdot, \lambda)$ and $V_0 = V_0(\cdot, \lambda)$ given by*

$$
\begin{aligned}
U_0(x) &= \exp\left(c\frac{x-a}{\omega}\right) \cdot \begin{pmatrix} u_0(x) \\ (p_0 u_0')(x) \end{pmatrix}, \\
V_0(x) &= \exp\left(-c\frac{x-a}{\omega}\right) \cdot \begin{pmatrix} v_0(x) \\ (p_0 v_0')(x) \end{pmatrix}
\end{aligned}
\tag{2.7}
$$

on (a, ∞) have the following property:

(i) *If $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H_0)$, then U_0 and V_0 are both ω-periodic and bounded on (a, ∞), where $\operatorname{Re} c > 0$.*

(ii) *If λ is an interior point of $\sigma_{\mathrm{ess}}(H_0)$, then U_0 and V_0 are both ω-periodic and bounded on (a, ∞), where $\operatorname{Re} c = 0$. In particular, $|u_0|$ and $|v_0|$ are ω-periodic and bounded on (a, ∞).*

(iii) *If λ is a boundary point of $\sigma_{\mathrm{ess}}(H_0)$, then U_0 is ω-periodic and bounded on (a, ∞), where $\operatorname{Re} c = 0$ and, in particular, $|u_0|$ is ω-periodic and bounded on (a, ∞). Furthermore, V_0 satisfies*

$$\|V_0(x)\|_{\mathbb{C}^2} \le C\left(1 + \frac{x-a}{\omega}\right) \tag{2.8}$$

on (a, ∞) for some positive constant C.

In the cases (i) and (iii) the solutions u_0 and v_0 can be chosen to be real-valued. Moreover, if $\lambda \in \sigma_{\mathrm{ess}}(H_0)$, then for every non-trivial solution of $(\tau_0 - \lambda)u = 0$ there exists a positive constant E such that

$$\int_{a+n\omega}^{a+(n+1)\omega} |u(t)|^2 r_0(t)\,\mathrm{d}t \ge E \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

Proof. Let $\mathcal{L}$ be the two-dimensional complex space of solutions of $(\tau_0 - \lambda)u = 0$. As the coefficients of τ_0 are ω-periodic, for every $f \in \mathcal{L}$ the function $f(\cdot + \omega)$ is again in $\mathcal{L}$. Now we identify the linear map $\mathcal{M} : \mathcal{L} \to \mathcal{L}$, $f \mapsto f(\cdot + \omega)$ with the matrix

$$M = \begin{pmatrix} \hat{u}(a+\omega) & \hat{v}(a+\omega) \\ (p_0\hat{u}')(a+\omega) & (p_0\hat{v}')(a+\omega) \end{pmatrix},$$

where $\hat{u}, \hat{v} \in \mathcal{L}$ are chosen such that $\hat{u}(a) = 1$, $(p_0\hat{u}')(a) = 0$ and $\hat{v}(a) = 0$, $(p_0\hat{v}')(a) = 1$. Since $\det M$ coincides with the Wronskian the spectrum is

$$\sigma(\mathcal{M}) = \sigma(M) = \{\mathrm{e}^{c}, \mathrm{e}^{-c}\}, \quad \text{where } c \in \mathbb{C}.$$

From now on fix the *Floquet exponent* c such that $\operatorname{Re} c \ge 0$. The eigenvalues $e^{\pm c}$ solve the quadratic equation $\det(M - z) = z^2 - Dz + 1 = 0$, where the Hill discriminant $D := D(\lambda) = \hat{u}(a+\omega) + (p_0\hat{v}')(a+\omega)$ is real. Therefore,

$$\mathrm{e}^{\pm c} = \frac{D}{2} \pm \sqrt{\frac{D^2}{4} - 1} \quad \text{or} \quad \mathrm{e}^{\pm c} = \frac{D}{2} \mp \sqrt{\frac{D^2}{4} - 1}. \tag{2.10}$$

Recall that by [15, Chapter 12 and Appendix] and [16, Chapter 16]

$$\sigma_{\mathrm{ess}}(H_0) = \{\lambda \in \mathbb{R} : |D(\lambda)| \le 2\} \quad \text{and} \quad \partial\sigma_{\mathrm{ess}}(H_0) = \{\lambda \in \mathbb{R} : |D(\lambda)| = 2\}. \tag{2.11}$$

(i) For $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H_0)$ we have $|D| > 2$ and hence $\mathrm{e}^c \neq \mathrm{e}^{-c}$ are both real by (2.10), which leads to $\operatorname{Re} c > 0$. As $\mathcal{M}$ has two distinct eigenvalues, we find corresponding eigenvectors $u_0, v_0 \in \mathcal{L}$ satisfying

$$u_0(x+\omega) = (\mathcal{M}u_0)(x) = \mathrm{e}^{-c} u_0(x), \qquad (p_0 u_0')(x+\omega) = \mathrm{e}^{-c}(p_0 u_0')(x), \tag{2.12}$$

$$v_0(x+\omega) = (\mathcal{M}v_0)(x) = \mathrm{e}^{c} v_0(x), \qquad (p_0 v_0')(x+\omega) = \mathrm{e}^{c}(p_0 v_0')(x), \tag{2.13}$$

on (a,∞), where the equalities in (2.12) and (2.13) for the derivatives follow from the periodicity of p_0. From (2.12) and (2.13) one also sees that the functions U_0 and V_0 defined in (2.7) are both ω-periodic, and hence also bounded. This completes the proof of (i).

(ii) For an interior point λ of $\sigma_{\mathrm{ess}}(H_0)$ we have $|D| < 2$ by (2.11), and hence e^c and e^{-c} are non-real and complex conjugates of each other, which yields $\operatorname{Re} c = 0$. As in the proof of (i) $\mathcal{M}$ has a pair of distinct eigenvalues and we find corresponding eigenvectors $u_0, v_0 \in \mathcal{L}$ satisfying (2.12), (2.13), which shows the periodicity of the U_0 and V_0 given in (2.7) and finishes the proof of (ii).

(iii) For $\lambda \in \partial\sigma_{\mathrm{ess}}(H_0)$ we have $|D| = 2$ and hence $\mathrm{e}^c = \mathrm{e}^{-c} = D/2 \in \{-1, 1\}$ by (2.10), and therefore $\operatorname{Re} c = 0$. Again, we find $u_0 \in \mathcal{L}$ such that (2.12) holds and this shows the periodicity of the function U_0 defined in (2.7). If the geometric multiplicity of $\mathrm{e}^c = \mathrm{e}^{-c}$ is two, then there is a second linearly independent solution $v_0 \in \mathcal{L}$ which satisfies (2.13). In this case the function V_0 in (2.7) is ω-periodic and the estimate (2.8) holds for $C = \sup_{x \in [a, a+\omega]} \|V_0(x)\|_{\mathbb{C}^2}$. Otherwise, if the geometric multiplicity of $\mathrm{e}^c = \mathrm{e}^{-c}$ is one, then there is a Jordan chain of length two, that is, there exists $v_0 \in \mathcal{L}$ with $\mathcal{M}v_0 = \mathrm{e}^c v_0 + u_0$. One has

$$v_0(x+\omega) = \mathrm{e}^c v_0(x) + u_0(x), \quad (p_0 v_0')(x+\omega) = \mathrm{e}^c (p_0 v_0')(x) + (p_0 u_0')(x) \tag{2.14}$$

for all $x \in (a, \infty)$. Now consider

$$V_0(x) := \exp\left(-c\frac{x-a}{\omega}\right) \cdot \begin{pmatrix} v_0(x) \\ (p_0 v_0')(x) \end{pmatrix},$$

as in (2.7) and recall that $\operatorname{Re} c = 0$. With (2.14) we have

$$\|V_0(x+\omega)\|_{\mathbb{C}^2} = \left\| \begin{pmatrix} \mathrm{e}^c v_0(x) + u_0(x) \\ \mathrm{e}^c (p_0 v_0')(x) + (p_0 u_0')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \leq \|V_0(x)\|_{\mathbb{C}^2} + \|U_0(x)\|_{\mathbb{C}^2}. \tag{2.15}$$

Let $x \in (a, \infty)$ and $k \in \mathbb{N}$ with $k \leq (x-a)/\omega < k+1$. Then (2.15) and the periodicity of U_0 give successively

$$\begin{aligned}
\|V_0(x)\|_{\mathbb{C}^2} &\leq \|V_0(x-k\omega)\|_{\mathbb{C}^2} + k\|U_0(x-k\omega)\|_{\mathbb{C}^2} \\
&\leq \|V_0(x-k\omega)\|_{\mathbb{C}^2} + \frac{x-a}{\omega}\|U_0(x-k\omega)\|_{\mathbb{C}^2} \\
&\leq \sup_{t \in [a, a+\omega]} \big(\|V_0(t)\|_{\mathbb{C}^2} + \|U_0(t)\|_{\mathbb{C}^2}\big) \cdot \left(1 + \frac{x-a}{\omega}\right).
\end{aligned}$$

This shows (iii).

Since in the cases (i) and (iii) the spectrum of $\mathcal{M}$ is real, $\mathcal{M}$ can be regarded as a mapping in the real space of real-valued solutions of $(\tau_0 - \lambda)u = 0$ instead of the complex space $\mathcal{L}$. Hence, u_0 and v_0 can be chosen as real-valued solutions. Finally, to show (2.9), consider $\lambda \in \sigma_{\mathrm{ess}}(H_0)$ and let u_0, v_0 be as in (ii) or (iii). Choose $d_1 \in \mathbb{C}$ such that $w_0 := d_1 u_0 + v_0$ is orthogonal to u_0 in $L^2((a, a+\omega); r_0)$. We have $\mathcal{M}u_0 = \mathrm{e}^{-c} u_0$ and $\mathcal{M}v_0 = \mathrm{e}^c v_0 + d_0 u_0$, where $d_0 \in \{0, 1\}$. Thus,

$$\mathcal{M}w_0 = \big(\mathrm{e}^{-c} d_1 + d_0\big) u_0 + \mathrm{e}^c v_0 = \big(\mathrm{e}^{-c} d_1 + d_0 - \mathrm{e}^c d_1\big) u_0 + \mathrm{e}^c w_0$$

and successively for all $n \in \mathbb{N}$

$$\mathcal{M}^n w_0 = \gamma_n u_0 + \mathrm{e}^{cn} w_0, \quad \text{where } \gamma_n \in \mathbb{C}.$$

We consider a non-trivial linear combination $\alpha u_0 + \beta w_0$, where $\alpha, \beta \in \mathbb{C}$. Note that by (2.12) $u_0(t + n\omega) = (\mathcal{M}^n u_0)(t) = \mathrm{e}^{-nc} u_0(t)$ for $t \in [a, \infty)$ and $n \in \mathbb{N}$. Recall also that $\operatorname{Re} c = 0$. If $\beta = 0$, then

$$\int_{a+n\omega}^{a+(n+1)\omega} |\alpha u_0(t)|^2 r_0(t) \, \mathrm{d}t = \int_a^{a+\omega} |\alpha u_0(t)|^2 r_0(t) \, \mathrm{d}t > 0$$

for all $n \in \mathbb{N}$. Otherwise, if $\beta \neq 0$, then

$$\begin{aligned} &\int_{a+n\omega}^{a+(n+1)\omega} |\alpha u_0(t) + \beta w_0(t)|^2 r_0(t) \, \mathrm{d}t \\ &\qquad = \int_a^{a+\omega} |\alpha(\mathcal{M}^n u_0)(t) + \beta(\mathcal{M}^n w_0)(t)|^2 r_0(t) \, \mathrm{d}t \\ &\qquad = \int_a^{a+\omega} |(\alpha \mathrm{e}^{-cn} + \beta \gamma_n) u_0(t) + \beta \mathrm{e}^{cn} w_0(t)|^2 r_0(t) \, \mathrm{d}t \\ &\qquad \geq \int_a^{a+\omega} |\beta w_0(t)|^2 r_0(t) \, \mathrm{d}t > 0 \end{aligned}$$

for all $n \in \mathbb{N}$. In both cases we conclude (2.9) and Lemma 2.6 is shown. □

The solution's asymptotics are basically preserved under L^1-perturbations of τ_0 with respect to its coefficients. This is the content of the next lemma.

Lemma 2.7. *Let $\lambda \in \mathbb{R}$, assume that (2.2) holds and let u_0, v_0 and c be as in Lemma 2.6. Then there exist linearly independent solutions $u_1 = u_1(\cdot, \lambda)$ and $v_1 = v_1(\cdot, \lambda)$ of $(\tau_1 - \lambda)u = 0$ such that the following holds:*

(i) *If $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H_0)$, that is, $\operatorname{Re} c > 0$, then*

$$\exp\left(\operatorname{Re} c \frac{x-a}{\omega}\right) \cdot \left\| \begin{pmatrix} u_1(x) \\ (p_1 u_1')(x) \end{pmatrix} - \begin{pmatrix} u_0(x) \\ (p_0 u_0')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to \infty \tag{2.16}$$

and

$$\begin{aligned} \left\| \begin{pmatrix} u_1(x) \\ (p_1 u_1')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} &\leq C \exp\left(-\operatorname{Re} c \frac{x-a}{\omega}\right), \\ \left\| \begin{pmatrix} v_1(x) \\ (p_1 v_1')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} &\leq C \exp\left(\operatorname{Re} c \frac{x-a}{\omega}\right) \end{aligned} \tag{2.17}$$

on (a, ∞), where $C = C(\lambda)$ is a positive constant. In particular, u_1 is bounded on (a, ∞).

(ii) *If λ is an interior point of $\sigma_{\mathrm{ess}}(H_0)$, that is, $\operatorname{Re} c = 0$, then (2.16) and (2.17) hold on (a, ∞), and*

$$\left\| \begin{pmatrix} v_1(x) \\ (p_1 v_1')(x) \end{pmatrix} - \begin{pmatrix} v_0(x) \\ (p_0 v_0')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to \infty. \tag{2.18}$$

In particular, u_1 and v_1 are bounded on (a, ∞).

(iii) *If λ is a boundary point of $\sigma_{\mathrm{ess}}(H_0)$, that is, $\operatorname{Re} c = 0$, and (2.5) (and hence also (2.2)) holds, then u_1 satisfies (2.16) and the first inequality in (2.17) on (a, ∞). In particular, u_1 is bounded on (a, ∞). If (2.6) (and hence also (2.2) and (2.5)) holds, then v_1 satisfies (2.18).*

The solutions in (i) and (iii) can be chosen to be real-valued.

Proof. Let $\lambda \in \mathbb{R}$. We consider the systems $\phi' = A\phi$ and $\xi' = (A+B)\xi$ corresponding to $(\tau_0 - \lambda)u = 0$ and $(\tau_1 - \lambda)u = 0$, respectively, where

$$A = \begin{pmatrix} 0 & \frac{1}{p_0} \\ q_0 - \lambda r_0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \frac{1}{p_1} - \frac{1}{p_0} \\ (q_1 - q_0) - \lambda(r_1 - r_0) & 0 \end{pmatrix}.$$

From (2.2) we obtain $\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1(a,\infty)$. With u_0 and v_0 from Lemma 2.6 we consider the fundamental solution Φ of the system $\phi' = A\phi$ given by

$$\Phi(x) = \begin{pmatrix} u_0(x) & v_0(x) \\ (p_0 u_0')(x) & (p_0 v_0')(x) \end{pmatrix}, \quad x \in (a,\infty), \tag{2.19}$$

so that

$$\big(\Phi(t)\big)^{-1} = \frac{1}{W(u_0,v_0)} \begin{pmatrix} (p_0 v_0')(t) & -v_0(t) \\ -(p_0 u_0')(t) & u_0(t) \end{pmatrix}, \quad t \in (a,\infty),$$

where W is the Wronskian. With (2.7) in Lemma 2.6 we estimate for all $x, t \in [a,\infty)$

$$\|\Phi(x)(\Phi(t))^{-1}\|_{\mathbb{C}^{2\times 2}} \leq \tilde{E} e^{\operatorname{Re} c \frac{t-x}{\omega}} \|U_0(x)\|_{\mathbb{C}^2} \|V_0(t)\|_{\mathbb{C}^2} + \tilde{E} e^{\operatorname{Re} c \frac{x-t}{\omega}} \|U_0(t)\|_{\mathbb{C}^2} \|V_0(x)\|_{\mathbb{C}^2} \tag{2.20}$$

where $\tilde{E}$ is a suitable positive constant.

We show (i) and (ii). In this case, $\operatorname{Re} c \geq 0$ and U_0, V_0 are bounded. We consider the Banach space $\mathcal{B}$ of all continuous $\mathbb{C}^2$-valued functions with exponential decay of order $-\operatorname{Re} c/\omega$, that is,

$$\mathcal{B} := \big\{\xi : [a,\infty) \to \mathbb{C}^2 \text{ continuous} : \|\xi(x)\|_{\mathbb{C}^2} \leq \gamma e^{-\operatorname{Re} c \frac{x}{\omega}} \text{ for some } \gamma \geq 0 \text{ on } [a,\infty)\big\}$$

and the corresponding norm

$$\|\xi\|_{\mathcal{B}} := \sup_{x \in [a,\infty)} e^{\operatorname{Re} c \frac{x-a}{\omega}} \|\xi(x)\|_{\mathbb{C}^2} < \infty.$$

For $\xi \in \mathcal{B}$ we define

$$(T\xi)(x) := -\Phi(x) \int_x^\infty (\Phi(t))^{-1} B(t)\xi(t)\,\mathrm{d}t, \quad x \in [a,\infty). \tag{2.21}$$

The integral in (2.21) converges. Indeed, the estimate in (2.20) yields

$$\|\Phi(x)(\Phi(t))^{-1}\|_{\mathbb{C}^{2\times 2}} \leq E e^{\operatorname{Re} c \frac{t-x}{\omega}} \tag{2.22}$$

for $a \leq x \leq t < \infty$, where E is a suitable positive constant. Then (2.21) with (2.22) give

$$\begin{aligned} e^{\operatorname{Re} c \frac{x-a}{\omega}} \|(T\xi)(x)\|_{\mathbb{C}^2} &\leq e^{\operatorname{Re} c \frac{x-a}{\omega}} \int_x^\infty E e^{\operatorname{Re} c \frac{t-x}{\omega}} \|B(t)\|_{\mathbb{C}^{2\times 2}} \|\xi(t)\|_{\mathbb{C}^2} \mathrm{d}t \\ &\leq \|\xi\|_{\mathcal{B}}\, E \int_x^\infty \|B(t)\|_{\mathbb{C}^{2\times 2}} \mathrm{d}t < \infty \end{aligned} \tag{2.23}$$

and hence the integral in (2.21) exists. Moreover, we also conclude that $T\xi \in \mathcal{B}$ and T is a bounded everywhere defined operator in $\mathcal{B}$.

We claim that for $n \in \mathbb{N}$ the estimate

$$\|(T^n\xi)(x)\|_{\mathbb{C}^2} \leq e^{-\operatorname{Re} c \frac{x-a}{\omega}} \|\xi\|_{\mathcal{B}} \frac{1}{n!} \left(E \int_x^\infty \|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^n, \quad x \in [a,\infty), \tag{2.24}$$

holds. In fact, for $n = 1$ this is true by (2.23). Now assume that (2.24) holds for some $n \in \mathbb{N}$. We set $G(t) := \frac{1}{n+1}\left(E\int_t^\infty \|B(s)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}s\right)^{n+1}$ and compute

$$\begin{aligned}
\|(T^{n+1}\xi)(x)\|_{\mathbb{C}^2} &\le \int_x^\infty \|\Phi(x)(\Phi(t))^{-1}\|_{\mathbb{C}^{2\times 2}}\|B(t)\|_{\mathbb{C}^{2\times 2}}\|(T^n\xi)(t)\|_{\mathbb{C}^2}\mathrm{d}t\\
&\le \mathrm{e}^{-\operatorname{Re} c\frac{x-a}{\omega}}\int_x^\infty E\|B(t)\|_{\mathbb{C}^{2\times 2}}\|\xi\|_{\mathcal{B}}\frac{1}{n!}\left(E\int_t^\infty \|B(s)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}s\right)^n\mathrm{d}t\\
&= \mathrm{e}^{-\operatorname{Re} c\frac{x-a}{\omega}}\|\xi\|_{\mathcal{B}}\frac{1}{n!}\int_x^\infty -G'(t)\mathrm{d}t\\
&= \mathrm{e}^{-\operatorname{Re} c\frac{x-a}{\omega}}\|\xi\|_{\mathcal{B}}\frac{1}{(n+1)!}\left(E\int_x^\infty \|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^{n+1}
\end{aligned}$$

which shows (2.24) for any $n \in \mathbb{N}$. Hence,

$$\|T^n\xi\|_{\mathcal{B}} \le \|\xi\|_{\mathcal{B}}\frac{1}{n!}\left(E\int_a^\infty \|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^n$$

and the Neumann series $(I-T)^{-1} = \sum_{n\in\mathbb{N}} T^n$ converges in the operator norm induced by $\|\cdot\|_{\mathcal{B}}$. Observe that for a solution $\phi \in \mathcal{B}$ of $\phi' = A\phi$ the function $\xi := (I-T)^{-1}\phi \in \mathcal{B}$ satisfies $\xi' = (A+B)\xi$ since

$$\xi = T\xi + \phi \tag{2.25}$$

yields

$$\xi' = \Phi'\Phi^{-1}T\xi + B\xi + \phi' = A(T\xi+\phi) + B\xi = (A+B)\xi. \tag{2.26}$$

Furthermore, from (2.25) and (2.23) we also conclude

$$\mathrm{e}^{\operatorname{Re} c\frac{x-a}{\omega}}\|\phi(x)-\xi(x)\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to \infty. \tag{2.27}$$

Now let us consider the continuous function $(u_0, p_0u_0')^\top : [a,\infty) \to \mathbb{C}^2$. According to Lemma 2.6 (i)–(ii) we have $(u_0, p_0u_0')^\top \in \mathcal{B}$. From the above considerations we see that $(I-T)^{-1}(u_0, p_0u_0')^\top$ is a solution of $\xi' = (A+B)\xi$ and hence

$$\begin{pmatrix} u_1 \\ p_1u_1' \end{pmatrix} := (I-T)^{-1}\begin{pmatrix} u_0 \\ p_0u_0' \end{pmatrix} \in \mathcal{B}$$

gives a solution u_1 of $(\tau_1-\lambda)u = 0$ such that the assertions in (i) and (ii) hold for u_1; note that (2.27) implies (2.16) and $(u_1, p_1u_1')^\top \in \mathcal{B}$ shows the first inequality in (2.17). Observe, that if λ is an interior point of $\sigma_{\mathrm{ess}}(H_0)$ then also $(v_0, p_0v_0')^\top \in \mathcal{B}$ by Lemma 2.6 (ii) as $\operatorname{Re} c = 0$. Again it follows that

$$\begin{pmatrix} v_1 \\ p_1v_1' \end{pmatrix} := (I-T)^{-1}\begin{pmatrix} v_0 \\ p_0v_0' \end{pmatrix} \in \mathcal{B}$$

gives a solution v_1 of $(\tau_1-\lambda)u = 0$ and (2.18) follows from (2.27). Thus we have shown (ii) and it remains to check in (i) the second inequality in (2.17). In fact, for any solution v_1 of $(\tau_1-\lambda)u = 0$ and $\xi = (v_1, p_1v_1')^\top$ one has

$$\xi(x) = \Phi(x)\left(\left(\Phi(a)\right)^{-1}\xi(a) + \int_a^x \left(\Phi(t)\right)^{-1}B(t)\xi(t)\,\mathrm{d}t\right).$$

From (2.20) we obtain $\|\Phi(x)(\Phi(t))^{-1}\|_{\mathbb{C}^{2\times 2}} \leq E\mathrm{e}^{\operatorname{Re} c\frac{x-t}{\omega}}$ for $a \leq t \leq x < \infty$ (cf. (2.22)) with some $E > 0$. Hence,

$$\mathrm{e}^{-\operatorname{Re} c\frac{x-a}{\omega}}\|\xi(x)\|_{\mathbb{C}^2} \leq E\|\xi(a)\|_{\mathbb{C}^2} + E\int_a^x \|B(t)\|_{\mathbb{C}^{2\times 2}}\left(\mathrm{e}^{-\operatorname{Re} c\frac{t-a}{\omega}}\|\xi(t)\|_{\mathbb{C}^2}\right)\mathrm{d}t$$

for all $x \in [a,\infty)$. Now Gronwall's inequality yields

$$\|\xi(x)\|_{\mathbb{C}^2} \leq \mathrm{e}^{\operatorname{Re} c\frac{x-a}{\omega}} E\|\xi(a)\|_{\mathbb{C}^2}\mathrm{e}^{E\int_a^x\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t},$$

and hence the second inequality in (2.17) holds for any solution v_1 of $(\tau_1-\lambda)u = 0$. This completes the proof of (i) and (ii).

We prove (iii). In the case $\lambda \in \partial\sigma_{\mathrm{ess}}(H_0)$ Lemma 2.6 (iii) implies $\operatorname{Re} c = 0$ and the Banach space $\mathcal{B}$ from above is the usual space of bounded continuous functions. Let $\phi \in \mathcal{B}$ and let T be as in (2.21). From Lemma 2.6 (iii) and (2.20) we obtain

$$\|\Phi(x)(\Phi(t))^{-1}\|_{\mathbb{C}^{2\times 2}} \leq E\left(1+\frac{t-a}{\omega}\right) \tag{2.28}$$

for $a \leq x \leq t < \infty$ and hence

$$\|(T\phi)(x)\|_{\mathbb{C}^2} \leq \|\phi\|_{\mathcal{B}}\, E\int_x^\infty \left(1+\frac{t-a}{\omega}\right)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t, \tag{2.29}$$

where the integral converges since $(1+|\cdot|)\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1(a,\infty)$ by (2.5). In the same way as in the proof of (i) and (ii) one verifies with $G(t)$ replaced by $H(t) = \frac{1}{n+1}\left(E\int_t^\infty(1+\frac{s-a}{\omega})\|B(s)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}s\right)^{n+1}$ that

$$\|(T^n\phi)(x)\|_{\mathbb{C}^2} \leq \|\phi\|_{\mathcal{B}}\frac{1}{n!}\left(E\int_x^\infty\left(1+\frac{t-a}{\omega}\right)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^n$$

and

$$\|(T^n\phi)\|_{\mathcal{B}} \leq \|\phi\|_{\mathcal{B}}\frac{1}{n!}\left(E\int_a^\infty\left(1+\frac{t-a}{\omega}\right)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^n$$

hold for all $n \in \mathbb{N}$ and $x \in [a,\infty)$. As above it follows that $(I-T)^{-1}$ is an everywhere defined bounded operator in $\mathcal{B}$ and for a solution $\phi \in \mathcal{B}$ of $\phi' = A\phi$ the function $\xi = (I-T)^{-1}\phi \in \mathcal{B}$ satisfies (2.25) and (2.26). Hence it follows from (2.29) that (2.27) holds with $\operatorname{Re} c = 0$. Now consider $(u_0, p_0u_0')^\top$, which is in $\mathcal{B}$ by Lemma 2.6 (iii), and set

$$\begin{pmatrix} u_1 \\ p_1u_1' \end{pmatrix} := (I-T)^{-1}\begin{pmatrix} u_0 \\ p_0u_0' \end{pmatrix} \in \mathcal{B}. \tag{2.30}$$

Then u_1 is a solution of $(\tau_1-\lambda)u = 0$ and the assertions for u_1 in (iii) follow.

Now assume that the integrability condition (2.6) (and hence also (2.2) and (2.5)) holds. Then $(1+|\cdot|^2)\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1(a,\infty)$ and for continuous functions $\xi : [a,\infty) \to \mathbb{C}^2$ such that

$$C_\xi := \sup_{x\in[a,\infty)}\left(1+\tfrac{x-a}{\omega}\right)^{-1}\|\xi(x)\|_{\mathbb{C}^2} < \infty \tag{2.31}$$

we can consider the integral (2.21), where we shall use the notation $\widetilde{T}$ to distinguish from the operator T acting in the Banach space $\mathcal{B}$. In fact, by (2.28) we have

$$\|(\widetilde{T}\xi)(x)\|_{\mathbb{C}^2} \leq E\,C_\xi\int_x^\infty\left(1+\frac{t-a}{\omega}\right)^2\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t \tag{2.32}$$

for $x \in (a, \infty)$ and hence $\widetilde{T}\xi \in \mathcal{B}$. Now let $\phi = (v_0, p_0 v_0')^\top$ and observe that by Lemma 2.6 (iii) ϕ satisfies an estimate of the form (2.31). The function $\xi := (I - T)^{-1}\widetilde{T}\phi + \phi$ also satisfies (2.31) and $\xi - \phi = (I - T)^{-1}\widetilde{T}\phi \in \mathcal{B}$. Hence,

$$\widetilde{T}\phi = (I - T)(\xi - \phi) = (\xi - \phi) - \widetilde{T}(\xi - \phi) = \xi - \phi - \widetilde{T}\xi + \widetilde{T}\phi,$$

which implies

$$\xi = \phi + \widetilde{T}\xi. \tag{2.33}$$

As in (2.26) we see that ξ solves $\xi' = (A+B)\xi$ and hence $\xi = (v_1, p_1 v_1')^\top$ with some solution v_1 of $(\tau_1 - \lambda)u = 0$. From (2.32) and (2.33) we obtain $\|\phi(x) - \xi(x)\|_{\mathbb{C}^2} \to 0$ as $x \to \infty$, which shows (2.18). To see that v_1 and u_1 in the present situation are linearly independent assume the contrary. Then also $(v_1, p_1 v_1')^\top$ and $(u_1, p_1 u_1')^\top$ are multiples of each other and hence $(v_1, p_1 v_1')^\top \in \mathcal{B}$. But then also

$$(I - \widetilde{T})(v_1, p_1 v_1')^\top = (I - T)(v_1, p_1 v_1')^\top = (v_0, p_0 v_0')^\top$$

and $(I - T)(u_1, p_1 u_1')^\top = (u_0, p_0 u_0')^\top$ (see (2.30)) are multiples of each other; a contradiction.

Note that in the cases (i) and (iii) the solutions u_0 and v_0 from Lemma 2.6 can be chosen to be real-valued. Then Φ in (2.19) has values in $\mathbb{R}^{2\times 2}$ and the solution u_1 and v_1 in (i) and (iii) constructed via T in (2.21) are also real-valued. □

Proof of Theorem 2.1. For $\lambda \in \mathbb{R}$ let $c = c(\lambda)$ and $u_j = u_j(\cdot, \lambda), v_j(\cdot, \lambda)$, $j = 0, 1$, be as in Lemma 2.6 and Lemma 2.7. The proof is divided into four steps.

Step 1. Let λ be an arbitrary element of the non-empty interior of $\sigma_{\mathrm{ess}}(H_0)$, that is, $\operatorname{Re} c = 0$ by Lemma 2.6 (ii). We show that for every nonzero solution w_1 of $(\tau_1 - \lambda)u = 0$ there exist positive constants E_1 and E_2 such that

$$E_1 \le \int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 r_1(t)\, \mathrm{d}t \le E_2 \tag{2.34}$$

holds for all sufficiently large $n \in \mathbb{N}$. Fix an arbitrary nontrivial linear combination $w_1 = \alpha u_1 + \beta v_1$, $\alpha, \beta \in \mathbb{C}$. For the same constants α and β let $w_0 = \alpha u_0 + \beta v_0$. From Lemma 2.6 (ii) and the periodicity of U_0 and V_0 we obtain for $n \in \mathbb{N}$ and $t \in [a, \infty)$

$$u_0(t + n\omega) = \mathrm{e}^{-nc} u_0(t) \quad \text{and} \quad v_0(t + n\omega) = \mathrm{e}^{nc} v_0(t).$$

This, $|\mathrm{e}^{-nc}| = |\mathrm{e}^{nc}| = 1$, and the periodicity of r_0 imply

$$\begin{aligned}\int_{a+n\omega}^{a+(n+1)\omega} |w_0(t)|^2 r_0(t)\, \mathrm{d}t &\le 2 \int_{a+n\omega}^{a+(n+1)\omega} \big(|\alpha u_0(t)|^2 + |\beta v_0(t)|^2\big) r_0(t)\, \mathrm{d}t \\ &= 2 \int_a^{a+\omega} \big(|\alpha \mathrm{e}^{-nc} u_0(t)|^2 + |\beta \mathrm{e}^{nc} v_0(t)|^2\big) r_0(t)\, \mathrm{d}t,\end{aligned}$$

and hence together with (2.9) we conclude

$$E \le \int_{a+n\omega}^{a+(n+1)\omega} |w_0(t)|^2 r_0(t)\, \mathrm{d}t \le E'$$

for some $E, E' > 0$ and all $n \in \mathbb{N}$. Furthermore,

$$\begin{aligned}\big||w_1|^2 r_1 - |w_0|^2 r_0\big| &= \big||w_1|^2 (r_1 - r_0) + \big(|w_1|^2 - |w_0|^2\big) r_0\big| \\ &\le |w_1|^2 |r_1 - r_0| + |w_1 - w_0| \big(|w_1| + |w_0|\big) r_0\end{aligned} \tag{2.35}$$

holds pointwise a. e. on (a,∞). By Lemma 2.6 (ii) and Lemma 2.7 (ii) the solutions w_0, w_1 are bounded and

$$|w_1(x)-w_0(x)| \le |\alpha|\cdot|u_1(x)-u_0(x)|+|\beta|\cdot|v_1(x)-v_0(x)|\to 0, \quad \text{as } x\to\infty$$

by (2.16) and (2.18). Thus, (2.35) together with $r_1-r_0\in L^1(a,\infty)$ and the periodicity of r_0 imply the existence of $n_0\in\mathbb{N}$ such that

$$\left|\int_{a+n\omega}^{a+(n+1)\omega}|w_1(t)|^2 r_1(t)\,\mathrm{d}t-\int_{a+n\omega}^{a+(n+1)\omega}|w_0(t)|^2 r_0(t)\,\mathrm{d}t\right|\le\frac{E}{2}$$

for all $n\ge n_0$. Choosing $E_1=\frac{E}{2}$ and $E_2=E'+\frac{E}{2}$ shows (2.34) for all $n\ge n_0$.

As an immediate consequence, τ_1 is in the limit-point case at ∞ and no non-trivial solution of $(\tau_1-\lambda)u=0$ is in $L^2((a,\infty);r_1)$, and thus $\lambda\in\sigma_{\mathrm{ess}}(H_1)$; cf. [15, Theorem 11.5]. Since the essential spectra are closed sets we obtain

$$\sigma_{\mathrm{ess}}(H_0)\subset\sigma_{\mathrm{ess}}(H_1).$$

Step 2. Let λ be an arbitrary element of the non-empty interior of $\sigma_{\mathrm{ess}}(H_0)$. We prove now the statement on the absolute continuous spectrum of H_1. A non-trivial solution u of $(\tau_1-\lambda)u=0$ for real λ is called *sequentially subordinant* at ∞ with respect to another non-trivial solution v of $(\tau_1-\lambda)u=0$ if

$$\liminf_{x\to\infty}\frac{\int_a^x|u(t)|^2 r_1(t)\,\mathrm{d}t}{\int_a^x|v(t)|^2 r_1(t)\,\mathrm{d}t}=0,$$

see [14, Section 9.5] and also [12]. By (2.34) in the first step of proof above we see that for all interior points λ of $\sigma_{\mathrm{ess}}(H_1)$ no sequentially subordinate solution of $(\tau_1-\lambda)u=0$ exists. Standard subordinancy theory (cf. Theorem 9.27 together with the remark below in [14]) implies that the absolutely continuous spectrum of H_1 equals $\sigma_{\mathrm{ess}}(H_1)$ and the interior of $\sigma_{\mathrm{ess}}(H_1)$ is purely absolutely continuous.

Step 3. We proceed to prove the converse inclusion $\sigma_{\mathrm{ess}}(H_1)\subset\sigma_{\mathrm{ess}}(H_0)$. Suppose $\lambda\notin\sigma_{\mathrm{ess}}(H_0)$, that is, $\operatorname{Re} c>0$ by Lemma 2.6 (i). By Lemma 2.7 (i) there exist real-valued solutions u_1 and v_1. For $g\in L^2((a,\infty);r_1)$ set

$$(Sg)(x):=\frac{1}{W(u_1,v_1)}\int_a^\infty G(x,t)g(t)r_1(t)\,\mathrm{d}t,\quad G(x,t):=\begin{cases}u_1(x)v_1(t) & \text{if } a\le t\le x,\\ u_1(t)v_1(x) & \text{if } a\le x\le t,\end{cases}$$

that is

$$(Sg)(x)=\frac{1}{W(u_1,v_1)}\left(u_1(x)\int_a^x v_1(t)g(t)r_1(t)\,\mathrm{d}t+v_1(x)\int_x^\infty u_1(t)g(t)r_1(t)\,\mathrm{d}t\right), \tag{2.36}$$

where W stands again for the Wronskian. Define

$$E:=\sup_{n\in\mathbb{N}}\int_{a+n\omega}^{a+(n+1)\omega} r_1(t)\,\mathrm{d}t,$$

which is finite since $r_0-r_1\in L^1(a,\infty)$ and r_0 is periodic and locally integrable. Consider an arbitrary $x\in[a,\infty)$. By (2.17) in Lemma 2.7 (i)

$$\int_a^\infty|G(x,t)|r_1(t)\,\mathrm{d}t\le C^2\left(\int_a^x \mathrm{e}^{\operatorname{Re} c\frac{t-x}{\omega}}r_1(t)\,\mathrm{d}t+\int_x^\infty \mathrm{e}^{\operatorname{Re} c\frac{x-t}{\omega}}r_1(t)\,\mathrm{d}t\right).$$

Let $k \in \mathbb{N}$ with $k\omega + a \le x < (k+1)\omega + a$. We continue estimating

$$\int_a^\infty |G(x,t)| r_1(t)\,\mathrm{d}t \le C^2 \sum_{n=0}^{k} \mathrm{e}^{\operatorname{Re} c\cdot(1-n)} \int_{a+(k-n)\omega}^{a+(k+1-n)\omega} r_1(t)\,\mathrm{d}t$$

$$+ C^2 \sum_{n=0}^{\infty} \mathrm{e}^{\operatorname{Re} c\cdot(1-n)} \int_{a+(n+k)\omega}^{a+(n+1+k)\omega} r_1(t)\,\mathrm{d}t$$

$$\le 2C^2 E \sum_{n=0}^{\infty} \mathrm{e}^{\operatorname{Re} c\cdot(-n+1)} < \infty.$$

Due to the symmetry $G(x,t) = G(t,x)$ the same bound holds for $\int_a^\infty |G(x,t)| r_1(x)\,\mathrm{d}x$ evaluated at $t \in [a,\infty)$. As a consequence of the Schur criterion (see, e. g., [14, Lemma 0.32]) one obtains that S is a bounded operator in $L^2((a,\infty); r_1)$. For $g \in L^2((a,\infty); r_1)$ a straightforward calculation using (2.36) and $(\tau_1 - \lambda)u_1 = (\tau_1 - \lambda)v_1 = 0$ shows that Sg, $p_1(Sg)'$ are absolutely continuous on (a,∞), and that Sg solves the inhomogeneous differential equation $(\tau_1 - \lambda)u = g$. Thus, $\tau_1(Sg) = \lambda Sg + g \in L^2((a,\infty); r_1)$ and hence Sg is in the domain of the maximal operator associated to τ_1 in $L^2((a,\infty); r_1)$ and S is injective. Moreover, since u_1 and v_1 are real-valued it follows that S is self-adjoint, so that S^{-1} is a self-adjoint restriction of the maximal operator associated with $\tau_1 - \lambda$. In other words, S is the resolvent at λ of some self-adjoint realization of τ_1 and as all self-adjoint realizations of τ_1 have the same essential spectrum, we obtain

$$\lambda \notin \sigma_{\mathrm{ess}}(H_1).$$

Thus $\sigma_{\mathrm{ess}}(H_1) \subset \sigma_{\mathrm{ess}}(H_0)$ and together with the first step

$$\sigma_{\mathrm{ess}}(H_1) = \sigma_{\mathrm{ess}}(H_0).$$

Step 4. Recall that the periodic Sturm–Liouville operator H_0 is semibounded from below. Let $\lambda < \inf \sigma_{\mathrm{ess}}(H_0)$, that is, $\operatorname{Re} c > 0$ by Lemma 2.6 (i). It is no restriction to assume that the solutions u_0 and u_1 provided by Lemma 2.6 (i) and Lemma 2.7 (i) are real-valued. Since H_0 is semibounded from below the differential expression $\tau_0 - \lambda$ is non-oscillatory (see [15, Theorem 14.9]), that is, u_0 has at most finitely many zeros in (a,∞). Furthermore, Lemma 2.6 (i) implies that the function $\tilde{u}_0$ given by

$$\tilde{u}_0(x) = \mathrm{e}^{c\frac{x-a}{\omega}} u_0(x)$$

is ω-periodic. Therefore, the solution u_0 has no zeros and

$$\gamma := \inf_{t\in(a,\infty)} |\tilde{u}_0(t)| = \min_{t\in[a,a+\omega]} |\tilde{u}_0(t)| > 0.$$

Assume that H_1 is not semibounded from below. Then [15, Theorem 14.9] implies that the differential expression $\tau_1 - \lambda$ is oscillatory, and hence the solution u_1 of $(\tau_1 - \lambda)u = 0$ has infinitely many zeros $x_0 < x_1 < x_2 < \dots$ accumulating at ∞. Together with (2.16) we obtain

$$0 < \gamma \le |\tilde{u}_0(x_n)| = |\mathrm{e}^{c\frac{x_n-a}{\omega}} u_0(x_n)| = \mathrm{e}^{\operatorname{Re} c\frac{x_n-a}{\omega}} |u_0(x_n) - u_1(x_n)| \to 0 \quad \text{as } n \to \infty;$$

a contradiction. This shows the semiboundedness of H_1. $\square$

Proof of Theorem 2.3. Suppose that (2.5) (and hence also (2.2)) holds. We show that every gap of the essential spectrum of H_1 contains at most finitely many eigenvalues of H_1. The proof is similar as in Step 4 in the proof of Theorem 2.1, but instead of the zeros of solutions we consider the zeros of modified Wronskians. Let μ, $\lambda \in \mathbb{R}$ such that $\mu < \lambda$ with $\sigma_{\mathrm{ess}}(H_0) \cap (\mu, \lambda) = \sigma_{\mathrm{ess}}(H_1) \cap (\mu, \lambda) = \emptyset$. We have

$$\lambda,\, \mu \in \partial\sigma_{\mathrm{ess}}(H_0) \cup \big(\mathbb{R} \setminus \sigma_{\mathrm{ess}}(H_0)\big).$$

Let $c(\lambda)$, $c(\mu)$ be the Floquet exponents associated with $(\tau_0 - \lambda)u = 0$ and $(\tau_0 - \mu)u = 0$, respectively. For the real-valued solutions $u_j(\cdot, \lambda)$ and $u_j(\cdot, \mu)$, where $j = 0, 1$, provided by Lemma 2.6 (i), (iii) and Lemma 2.7 (i), (iii) we consider the modified Wronskians

$$W_j(x) := W(u_j(\cdot, \mu), u_j(\cdot, \lambda))(x) = \begin{pmatrix} u_j(x, \lambda) \\ p_j(x)u_j'(x, \lambda) \end{pmatrix}^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_j(x, \mu) \\ p_j(x)u_j'(x, \mu) \end{pmatrix}$$

Observe that

$$\widetilde{W_0}(x) := \exp\left(\big(c(\lambda) + c(\mu)\big)\frac{x - a}{\omega}\right) W_0(x) = \big(U_0(x, \lambda)\big)^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U_0(x, \mu), \tag{2.37}$$

where $U_0(\cdot, \lambda)$ and $U_0(\cdot, \mu)$ are ω-periodic functions given by (2.7) in Lemma 2.6. Therefore, the function $\widetilde{W_0}$ is ω-periodic. Since there is at most one simple eigenvalue of H_0 in (μ, λ) we conclude from [5, Theorem 7.5 (i)] that W_0 has at most finitely many zeros in (a, ∞). According to the periodicity of $\widetilde{W_0}$ together with (2.37), the modified Wronskian W_0 has no zeros and

$$\gamma := \inf_{t \in (a, \infty)} |\widetilde{W_0}(t)| = \min_{t \in [a, a+\omega]} |\widetilde{W_0}(t)| > 0.$$

The difference of W_0 and W_1 can be written as

$$\begin{aligned} W_0(x) - W_1(x) = {} & \left(\begin{pmatrix} u_0(x, \lambda) \\ (p_0(x)u_0'(x, \lambda) \end{pmatrix} - \begin{pmatrix} u_1(x, \lambda) \\ p_1(x)u_1'(x, \lambda) \end{pmatrix}\right)^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0(x, \mu) \\ p_0(x)u_0'(x, \mu) \end{pmatrix} \\ & + \begin{pmatrix} u_1(x, \lambda) \\ p_1(x)u_1'(x, \lambda) \end{pmatrix}^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} u_0(x, \mu) \\ p_0(x)u_0'(x, \mu) \end{pmatrix} - \begin{pmatrix} u_1(x, \mu) \\ p_1(x)u_1'(x, \mu) \end{pmatrix}\right). \end{aligned}$$

Combining this with Lemma 2.6 (i), (iii) and Lemma 2.7 (i), (iii) we conclude

$$\exp\left(\big(c(\lambda) + c(\mu)\big)\frac{x - a}{\omega}\right) \cdot (W_0(x) - W_1(x)) \to 0 \quad \text{as } x \to \infty. \tag{2.38}$$

Now assume that H_1 has infinitely many eigenvalues in (μ, λ). Then the modified Wronskian W_1 has infinitely many zeros $x_0 < x_1 < x_2 < \dots$ which necessarily accumulate at ∞; cf. [5, Theorem 7.5 (i)]. Then (2.38) implies

$$\begin{aligned} 0 < \gamma \le |\widetilde{W_0}(x_n)| &= |\exp\left(\big(c(\lambda) + c(\mu)\big)\frac{x_n - a}{\omega}\right) W_0(x_n)| \\ &= |\exp\left(\big(c(\lambda) + c(\mu)\big)\frac{x_n - a}{\omega}\right) \big(W_0(x_n) - W_1(x_n)\big)| \to 0 \quad \text{as } n \to \infty; \end{aligned}$$

a contradiction. Hence, $\dim \operatorname{ran}(P_{(\mu,\lambda)}(H_1)) < \infty$. $\square$

Proof of Theorem 2.4. Suppose that (2.6) (and hence also (2.2) and (2.5)) holds. We show that the boundary points of the essential spectrum of H_1 are no eigenvalues of H_1 and, therefore, $\sigma_{\mathrm{ess}}(H_1)$ is purely absolutely continuous. Let $\lambda \in \partial\sigma_{\mathrm{ess}}(H_1)$ and consider an arbitrary non-trivial linear combination $w_1 := \alpha u_1 + \beta v_1$, where $\alpha, \beta \in \mathbb{C}$. For the same coefficients α, β let $w_0 := \alpha u_0 + \beta v_0$ and observe that by Lemma 2.7 (iii)

$$w_1(x) - w_0(x) \to 0 \quad \text{and hence} \quad |w_1(x)|^2 - |w_0(x)|^2 \to 0 \quad \text{as } x \to \infty. \tag{2.39}$$

We estimate with (2.8) and the boundedness of $|u_0|$ from Lemma 2.6 (iii) for some $M > 0$ and all $t \in [a, \infty)$

$$|w_0(t)|^2 \le \left(|\alpha||u_0(t)| + |\beta| C \left(1 + \frac{t-a}{\omega} \right) \right)^2 \le M(1 + t^2),$$

and hence

$$|w_0(t)|^2 |r_1(t) - r_0(t)| \le M(1 + t^2)|r_1(t) - r_0(t)|. \tag{2.40}$$

Moreover,

$$\left| |w_1|^2 r_1 - |w_0|^2 r_0 \right| \le \left| |w_1|^2 - |w_0|^2 \right| |r_1 - r_0| + |w_0|^2 |r_1 - r_0| + \left| |w_1|^2 - |w_0|^2 \right| r_0 \tag{2.41}$$

holds pointwise a. e. on (a, ∞) and by (2.6) the functions $t \mapsto t^2 |r_1(t) - r_0(t)|$ and $t \mapsto |r_1(t) - r_0(t)|$ are in $L^1(a, \infty)$. Thus, (2.41) together with (2.39), (2.40), and the periodicity of r_0 imply the existence of $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\left| \int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 r_1(t) \, \mathrm{d}t - \int_{a+n\omega}^{a+(n+1)\omega} |w_0(t)|^2 r_0(t) \, \mathrm{d}t \right| \le \frac{E}{2},$$

where the constant E is from (2.9). This gives for all $n \ge n_0$

$$\int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 r_1(t) \, \mathrm{d}t \ge \frac{E}{2}.$$

Therefore, w_1 does not belong to $L^2((a, \infty); r_1)$, which shows that $\lambda \in \partial\sigma_{\mathrm{ess}}(H_1)$ is not an eigenvalue of H_1. □

Proof of Proposition 2.5. Suppose that (2.5) (and hence also (2.2)) holds and that r_1 satisfies $C_0 \le r_1(t) \le C_1$ for t in some neighbourhood of ∞ for some positive constants C_0, C_1. Let λ be a boundary point of $\sigma_{\mathrm{ess}}(H_1)$, let $u_1 = u_1(\cdot, \lambda)$ be the solution found in Lemma 2.7 (iii), and suppose $v_1 = v_1(\cdot, \lambda)$ were an eigenfunction. Then, by (2.16) and (2.9), u_1 and v_1 must be linearly independent and we can rescale v_1 such that the Wronskian with u_1 satisfies

$$1 = W(u_1, v_1) = u_1(p_1 v_1') - (p_1 u_1') v_1.$$

In particular, we obtain

$$\frac{1}{2} \le r_1 u_1^2 \frac{(p_1 v_1')^2}{r_1} + r_1 v_1^2 \frac{(p_1 u_1')^2}{r_1}$$

Now since v_1 is an eigenfunction, we have $r_1 v_1^2 \to 0$ (at least for some subsequence). Moreover, by (2.16) and our assumption on r_1 both $r_1 u_1^2$ and $(p_1 u_1')^2 / r_1$ are bounded. Finally, the assumption $\int_{t-1}^{t+1} |q_1(s)|^2 ds \le C_1$ together with the other assumptions on r_1 and p_1 ensure that the first integral on the right hand side of [12, Eq. (2.21) in Lemma 2.7] is bounded and hence this lemma implies $(p_1 v_1')^2 / r_1 \to 0$, which gives a contradiction. Thus, there is no square summable solution for λ. □

3. Proof of the main results

Proofs of Theorem 1.1–1.3. Our main results follow from a coupling argument and applications of Theorems 2.1, 2.3 and 2.4 and their counterparts on the half-line $(-\infty, a)$. More precisely, choose any self-adjoint realization $A_{0,-}$ and $A_{0,+}$ of τ_0 in $L^2((-\infty,a);r_0)$ and $L^2((a,\infty);r_0)$, respectively, and observe that the resolvent difference of A_0 and $A_{0,-} \oplus A_{0,+}$ is an operator of rank one or rank two. In particular, A_0 and $A_{0,-} \oplus A_{0,+}$ have the same essential spectrum, and the periodicity also implies $\sigma_{\mathrm{ess}}(A_{0,-}) = \sigma_{\mathrm{ess}}(A_{0,+})$.

Let $A_{1,-}$ and $A_{1,+}$ be self-adjoint realizations of τ_1 in $L^2((-\infty,a);r_1)$ and $L^2((a,\infty);r_1)$, respectively. It follows from Theorem 2.1 that $A_{1,\pm}$ are semibounded, $\sigma_{\mathrm{ess}}(A_{0,\pm}) = \sigma_{\mathrm{ess}}(A_{1,\pm})$, and hence $A_{1,-} \oplus A_{1,+}$ is semibounded and

$$\sigma_{\mathrm{ess}}(A_0) = \sigma_{\mathrm{ess}}(A_{0,-} \oplus A_{0,+}) = \sigma_{\mathrm{ess}}(A_{1,-} \oplus A_{1,+}).$$

As also the resolvent difference of A_1 and $A_{1,-} \oplus A_{1,+}$ is an operator of rank one or rank two we conclude that A_1 is semibounded and

$$\sigma_{\mathrm{ess}}(A_0) = \sigma_{\mathrm{ess}}(A_1).$$

In order to prove Theorem 1.1 it remains to show the statement on the absolutely continuous spectrum of A_1. Let λ be an interior point of $\sigma_{\mathrm{ess}}(A_1)$ and let u be a non-trivial solution of $(\tau_1 - \lambda)u = 0$. Step 2 of the proof of Theorem 2.1 shows that the restrictions of u onto $(-\infty,a)$ and (a,∞) are not sequentially subordinant at $\pm\infty$ and from [6, Theorem 2] we conclude that the spectrum of A_1 is purely absolutely continuous in the interior of the spectral bands. This completes the proof of Theorem 1.1. Arguing with the restrictions of u onto $(-\infty,a)$ and (a,∞) in the same way as in the proof of Theorem 2.4 we also conclude that the band edges are no eigenvalues of A_1 under the assumptions of Theorem 1.3. To conclude Theorem 1.2 note that by Theorem 2.3 each gap contains at most finitely many eigenvalues of $A_{1,-} \oplus A_{1,+}$. As the resolvent difference of A_1 and $A_{1,-} \oplus A_{1,+}$ is at most of rank two the number of eigenvalues of A_1 in each gap can increase by at most two, which shows Theorem 1.2. □

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