ON THE FINITENESS OF THE NUMBER OF EIGENVALUES OF JACOBI OPERATORS BELOW THE ESSENTIAL SPECTRUM

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ABSTRACT. We present a new oscillation criterion to determine whether the number of eigenvalues below the essential spectrum of a given Jacobi operator is finite or not. As an application we show that Kenser's criterion for Jacobi operators follows as a special case.

1. INTRODUCTION

The goal of the present paper is to determine whether the number of eigenvalues below the essential spectrum of the Jacobi operator on $\ell^2(\mathbb{N})$ associated with

$$(1.1) \qquad (\tau f)(n) = a(n)f(n+1) + a(n-1)f(n-1) - b(n)f(n),$$

where

$$(1.2) \qquad a(n) \in \mathbb{R}\backslash\{0\}, \quad b(n) \in \mathbb{R}, \quad n \in \mathbb{N},$$

is finite or not.

We will assume $a(n) < 0$ (which is no restriction by [15], Lemma 1.6). One of the main cases of interest is $a(n) = -1$ and one usually starts with the operator H_0 associated with $b_0(n) = 2$. The spectrum is given by $\sigma(H_0) = [0, 4]$. In particular, there are no eigenvalues below the essential spectrum. Perturbing b_0 we can add any finite number of eigenvalues even if our perturbation is of compact support. However, the question is, can we at least determine whether the number of eigenvalues is finite or not, by looking at the asymptotics of the perturbation? Moreover, what is the precise asymptotics separating the two cases?

The natural tool for investigating such questions is oscillation theory since finiteness of the number of eigenvalues is equivalent to the operator being nonoscillatory. This fact first appeared in [5]. The precise relation between the number of eigenvalues and the number of *nodes* was established only recently by one of us in [14].

In the case of Sturm-Liouville operators there is a famous theorem by Kneser [10] which gives a simple and beautiful answer to this question, with many subsequent extensions by others. The most recent one being by [4], who give a unified result containing all previously known ones as special cases.

Unfortunately, it is not possible to use the same proof in the discrete case. To understand why, let us first review the proof of Kneser's theorem in the Sturm-Liouville case. The key idea is that the equation

$$(1.3) \qquad \tau_0 = -\frac{d^2}{dx^2} + \frac{\mu}{x^2}$$

is of Euler type. Hence it is explicitly solvable with a fundamental system given by

$$(1.4) \qquad x^{\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}}}.$$

There are two cases to distinguish. If $\mu \geq -1/4$ all solutions are nonoscillatory. If $\mu < -1/4$ one has to take real/imaginary parts and all solutions are oscillatory. Hence a straightforward application of Sturm's comparison theorem between τ_0 and

$$(1.5) \qquad \tau = -\frac{d^2}{dx^2} + q(x)$$

yields

$$(1.6) \qquad \lim_{x \to \infty} \begin{matrix} \inf \\ \sup \end{matrix} \left(x^2 q(x) \right) \begin{matrix} > \\ < \end{matrix} -\frac{1}{4} \text{ implies } \begin{matrix} \text{nonoscillation} \\ \text{oscillation} \end{matrix} \text{ of } \tau \text{ near } \infty.$$

Since Sturm's comparison theorem is also available for Jacobi operators (see, e.g, [15], Lemma 4.4) it seems easy to generalize this result by considering the discrete Euler equation

$$(1.7) \qquad u(n+1) - 2u(n) + u(n-1) - \frac{\mu}{n(n-1)} u(n-1) = 0.$$

However, unfortunately, this equation is not symmetric! The corresponding results for this equation can be found as special cases in [1], Section 6.11. Thus a straightforward generalization is not possible.

One approach is to consider generalized ordinary differential expressions which contain difference equations as a special case. This can be found in [5] and [11] (see Section 2.2). As an alternative, we will prove a new oscillation criterion and show that Kneser's criterion follows as a special case.

For further (non-)oscillation criteria we refer to Hinton and Lewis [7] and Hooker and Patula [8] (see also [6], [9], [12], and [13]). For a different approach using Birman-Schwinger type arguments see [2], [3].

Our present paper was motivated by the work of Gesztesy and Ünal [4] mentioned earlier. In fact, it can be viewed as a discrete generalization of their results. However, again a straightforward generalization is not possible since their proofs also rely on explicit solvability of the involved equations.

2. Main results and applications

Before we can write down our main result, we need to fix some notation. Recall that τ is called oscillatory if one (and hence any) real-valued solution of $\tau u = 0$ has an infinite number of nodes, that is, points $n \in \mathbb{N}$, such that either

$$(2.1) \qquad u(n) = 0 \quad \text{or} \quad a(n)u(n)u(n+1) > 0.$$

In the special case $a(n) < 0$, $n \in \mathbb{N}$, a node of u is precisely a sign flip of u as one would expect. In the general case, however, one has to take the sign of $a(n)$ into account.

Recall that if $u_0(n) > 0$ solves

(2.2) $\qquad (\tau_0 u)(n) = a(n)u_0(n+1) + a(n-1)u_0(n-1) - b_0(n)u_0(n) = 0,$

then

(2.3) $\qquad \hat{u}_0(n) = u_0(n)Q_0(n), \quad Q_0(n) = \sum_{j=0}^{n-1} \frac{-1}{a(j)u_0(j)u_0(j+1)},$

is a second, linearly independent positive solution. A positive solution is called minimal if

(2.4) $\qquad\qquad\qquad\qquad \lim_{n\to\infty} Q_0(n) = \infty.$

Minimal solutions are unique up to a multiple. See [15], Section 2.3 for more information. With this notation our main result reads as follows:

Theorem 2.1. *Suppose $a(n) \in \mathbb{R}\backslash\{0\}$, $b(n), b_0(n) \in \mathbb{R}$ such that $a_0 < |a(n)| < A_0$ for some real constants $0 < a_0 < A_0$ and let u_0 be a non-decreasing minimal positive solution of $\tau_0 u_0 = 0$ (as defined in (2.2)). Introduce*

(2.5) $\qquad\qquad\qquad A(n) = \frac{2a(n-1)a(n+1)}{a(n-1)+a(n+1)}, \quad n \in \mathbb{N},$

and

(2.6) $\qquad\qquad\qquad Q_0(n) = \sum_{j=0}^{n-1} \frac{-1}{a(j)u_0(j)u_0(j+1)}, \quad n \in \mathbb{N}.$

Then τ is nonoscillatory if

(2.7) $\qquad\qquad \liminf_{n\to\infty} -A(n)u_0^4(n)Q_0^2(n)(b(n)-b_0(n)) > -\frac{1}{4}$

and oscillatory if

(2.8) $\qquad\qquad \limsup_{n\to\infty} -A(n)u_0^4(n)Q_0^2(n)(b(n)-b_0(n)) < -\frac{1}{4}.$

The proof will be given in Section 3 below.

As a first application, let us show how this result can be used to answer our question posed in the introduction. We choose

(2.9) $\qquad\qquad\qquad\qquad a(n) = -1, \quad b_0(n) = 2.$

Then we have

(2.10) $\qquad\qquad\qquad\quad u_0(n) = 1 \quad \text{and} \quad \hat{u}_0(n) = n$

and thus

(2.11) $\quad \lim_{n\to\infty} \begin{matrix} \inf \\ \sup \end{matrix} (n^2(b(n)-2)) \begin{matrix} > \\ < \end{matrix} -\frac{1}{4} \text{ implies } \begin{matrix} \text{nonoscillation} \\ \text{oscillation} \end{matrix} \text{ of } \tau \text{ near } \infty,$

which is the claimed generalization of Kneser's result. Clearly, the next question is what happens in the limiting case, where $\lim_{n\to\infty} n^2(b(n)-2) = -4^{-1}$? This can be answered by our result as well:

Recall the iterated logarithm

(2.12) $\qquad\qquad\qquad \ln_0(x) = x, \qquad \ln_k(x) = \ln_{k-1}(\ln(x)),$

where $\ln_k(x)$ is defined for $x > e_k$, with $e_1 = 0$, $e_k = e^{e_{k-1}}$.

Corollary 2.2. *Let*

(2.13) $$a(n) = -1, \qquad b_k(n) = 2 - \frac{1}{4} \sum_{j=0}^{k-1} \frac{1}{\prod_{\ell=0}^{j} \ln_\ell(n)^2}.$$

Then τ is nonoscillatory if

(2.14) $$\liminf_{n \to \infty} \Big(\prod_{j=0}^{k} \ln_j(n) \Big)^2 (b(n) - b_k(n)) > -\frac{1}{4}$$

and oscillatory if

(2.15) $$\limsup_{n \to \infty} \Big(\prod_{j=0}^{k} \ln_j(n) \Big)^2 (b(n) - b_k(n)) < -\frac{1}{4}.$$

Proof. To show how this follows from our result we consider

(2.16) $$u_k(n) = \sqrt{\prod_{j=0}^{k-1} \ln_j(n)},$$

which is a solution of $\tilde{\tau}_k$ associated with

(2.17) $$a(n) = -1, \qquad \tilde{b}_k(n) = \frac{u_k(n+1) + u_k(n-1)}{u_k(n)}.$$

To prove the claim it suffices to show

(2.18) $$\begin{aligned} \tilde{b}_k(n) &= b_k(n) + O(n^{-3}), \\ Q_k(n) &= \ln_k(n) + O(1) \end{aligned}$$

since the differences will not contribute to the limits from above.

To establish (2.18) we first recall the following formulas for the first and second derivative of $\ln_k(x)$:

(2.19) $$\begin{aligned} \ln_k'(x) &= \prod_{j=0}^{k-1} \frac{1}{\ln_j(x)}, \\ \ln_k''(x) &= -\ln_k'(x) \sum_{j=1}^{k} \ln_j'(x), \qquad x > \mathrm{e}_k. \end{aligned}$$

Now we can show (2.18). First of all we have

(2.20) $$Q_k(n) = \int^n \frac{dx}{u_k(x)^2} + O(1) = \ln_k(n) + O(1).$$

The second claim is a bit harder. We begin with

$$
\begin{aligned}
\frac{u_k(n \pm 1)}{u_k(n)} &= \left(\prod_{j=0}^{k-1} \frac{\ln_j(n \pm 1)}{\ln_j(n)}\right)^{1/2} = \prod_{j=0}^{k-1}\left(\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \frac{\ln_j^{(\ell)}(n)}{\ln_j(n)}\right)^{1/2} \\
&= \prod_{j=0}^{k-1}\left(1 \pm \frac{\ln_j'(n)}{\ln_j(n)} + \frac{1}{2}\frac{\ln_j''(n)}{\ln_j(n)} + O(n^{-3})\right)^{1/2} \\
&= \prod_{j=0}^{k-1}\left(1 \pm \frac{1}{2}\ln_{j+1}'(n) + \frac{1}{4}\left(\frac{\ln_j''(n)}{\ln_j(n)} - \frac{\ln_{j+1}'(n)^2}{2}\right) + O(n^{-3})\right) \\
&= 1 \pm \frac{1}{2}\sum_{j=0}^{k-1}\ln_{j+1}'(n) + \frac{1}{4}\sum_{j=0}^{k-1}\left(\frac{\ln_j''(n)}{\ln_j(n)} - \frac{\ln_{j+1}'(n)^2}{2}\right) + \\
&\quad + \frac{1}{4}\sum_{j=0}^{k-1}\ln_{j+1}'(n)\sum_{\ell=0}^{j-1}\ln_{\ell+1}'(n) + O(n^{-3}).
\end{aligned}
$$

$$
\tag{2.21} = 1 \pm \frac{1}{2}\sum_{j=0}^{k-1}\ln_{j+1}'(n) - \frac{1}{8}\sum_{j=0}^{k-1}\ln_{j+1}'(n)^2 + O(n^{-3}).
$$

Now combining both formulas we obtain the desired result

$$
\tag{2.22} \tilde{b}_k(n) = 2 - \frac{1}{4}\sum_{j=0}^{k-1}\ln_{j+1}'(n)^2 + O(n^{-3}) = b_k(n) + O(n^{-3}).
$$

$\square$

Another interesting example is the case

$$
\tag{2.23} b_0(n) = -a(n) - a(n-1).
$$

Again we can take $u_0(n) = 1$ to obtain

Corollary 2.3. *Let $a_0 \le |a(n)| \le A_0$ and abbreviate*

$$
\tag{2.24} A(n) = \frac{2a(n-1)a(n+1)}{a(n-1) + a(n+1)}, \qquad Q_0(n) = \sum_{j=0}^{n-1}\frac{-1}{a(j)}.
$$

Then τ is nonoscillatory if

$$
\tag{2.25} \liminf_{n\to\infty} -A(n)Q_0(n)^2(b(n) + a(n-1) + a(n)) > -\frac{1}{4}
$$

and oscillatory if

$$
\tag{2.26} \limsup_{n\to\infty} -A(n)Q_0(n)^2(b(n) + a(n-1) + a(n)) < -\frac{1}{4}.
$$

Of course one could take two arbitrary sequence $a(n) < 0$ and $u_0(n) > 0$ such that u_0 is non-decreasing and (2.4) is satisfied, compute $b_0(n) = -(a(n)u_0(n+1) + a(n-1)u_0(n-1))/u_0(n)$, and apply Theorem 2.1 to get a new (non)oscillation criterion.

3. Proof of Theorem 2.1

Suppose $a(n) \in \mathbb{R}\setminus\{0\}$, $b(n), b_0(n) \in \mathbb{R}$ such that

$$(3.1) \qquad a_0 \leq |a(n)| \leq A_0$$

for some real constants $0 < a_0 < A_0$ and let u_0 be a non-decreasing minimal positive solution of $\tau_0 u_0 = 0$ as in the previous section. Note that the corresponding second positive solution $\hat{u}_0(n)$ is increasing.

First we collect some basic facts which will be needed later on.

Lemma 3.1. *Let u_0 be a minimal positive non-decreasing solution, then we have*

$$(3.2) \qquad \lim_{n\to\infty} \frac{u_0(n+1)}{u_0(n)} = \lim_{n\to\infty} \frac{u_0(n)}{u_0(n-1)} = 1$$

and

$$(3.3) \qquad u_0(n)\hat{u}_0(n) = u_0^2(n)Q_0(n) \geq \frac{n}{A_0}.$$

Proof. Monotonicity of u_0 implies

$$(3.4) \qquad \frac{1}{u_0(j+1)^2} \leq \frac{1}{u_0(j)u_0(j+1)} \leq \frac{1}{u_0(j)^2}.$$

Summing the last expression from 0 to $n-1$ and subtracting the right side yields

$$(3.5) \qquad 0 \leq \sum_{j=0}^{n-1} \frac{1}{u_0(j)u_0(j+1)}\left(\frac{u_0(j+1)}{u_0(j)} - 1\right) \leq \frac{1}{u_0(0)^2} - \frac{1}{u_0(n)^2} \leq \frac{1}{u_0(0)^2}.$$

Since $u(n)$ is minimal,

$$(3.6) \qquad \sum_{j=0}^{n-1} \frac{1}{u_0(j)u_0(j+1)} \geq a_0 Q_0(n) \to \infty$$

implies the first result.

For the second claim we use

$$(3.7) \qquad Q_0(n) = \sum_{j=0}^{n-1} \frac{-1}{a(j)u_0(j)u_0(j+1)} \geq \frac{1}{A_0}\sum_{j=0}^{n-1}\frac{1}{u_0(j+1)^2} \geq \frac{n}{A_0 u_0(n)^2}$$

finishing the proof. $\square$

Our next goal is to find a suitable comparison equation. We do this by trying the ansatz

$$(3.8) \qquad u_1(n) = u_0(n)Q_0(n)^\alpha.$$

Then $u_1(n)$ satisfies

$$(3.9) \qquad \tau_1 u(n) = a(n)u_1(n+1) + a(n-1)u_1(n-1) + b_1(n)u_1(n) = 0$$

with $b_1(n)$ given by

$$\begin{aligned}
b_1(n) &= -\frac{a(n)u_0(n+1)}{u_0(n)}\left(1 - \frac{1}{a(n)u_0(n+1)^2 Q_0(n)}\right)^\alpha \\
(3.10) \qquad &\quad -\frac{a(n-1)u_0(n-1)}{u_0(n)}\left(1 + \frac{1}{a(n-1)u_0(n-1)^2 Q_0(n)}\right)^\alpha.
\end{aligned}$$

In order to get an oscillating comparison equation we need to admit $\alpha \in \mathbb{C}$. However, this will also render $b_1(n)$ complex and hence it will be of no use for us.

To overcome this problem we look at the asymptotic behavior of $b_1(n)$ for $n \to \infty$, which is given by

$$(3.11) \qquad b_1(n) = b_0(n) + \mu U(n) + O\left(\frac{1}{u_0^6(n)Q_0^3(n)}\right), \quad \mu = \alpha(\alpha - 1),$$

where

$$(3.12) \qquad U(n) = \frac{1}{2u_0^4(n)Q_0^2(n)} \left(\frac{-u_0(n)}{a(n+1)u_0(n+1)} + \frac{-u_0(n)}{a(n-1)u_0(n-1)}\right).$$

If $\alpha \in \mathbb{R}$ we can choose $b_1(n)$ directly as comparison potential to obtain that τ is nonoscillatory if

$$(3.13) \qquad \liminf_{n \to \infty} \frac{b(n) - b_0(n)}{b_1(n) - b_0(n)} > \mu.$$

Using the optimal value $\alpha = \frac{1}{2}$ plus the expansion from above we end up with

$$(3.14) \qquad \liminf_{n \to \infty} \frac{b(n) - b_0(n)}{U(n)} > -\frac{1}{4}.$$

This settles the first part of our theorem. Now we come to the harder one. As already noticed, in order to get an oscillating comparison equation we need to choose complex values for α. Our strategy is to choose $\alpha = \frac{1}{2} + i\varepsilon$ such that at least $\mu = -\frac{1}{4} - \varepsilon^2$ remains real and take $\tilde{b}_1(n) = b_0(n) + \mu U(n)$ as comparison equation. Of course we do not know the solutions of this equation, but our hope is that they are asymptotically given by the real/imaginary parts of

$$(3.15) \qquad u_1(n) = u_0(n)\sqrt{Q_0(n)}\left(\cos(\varepsilon \ln Q_0(n)) + i\sin(\varepsilon \ln Q_0(n))\right).$$

Hence if we can show that there are solutions $\tilde{u}_1$ of $\tilde{\tau}_1 \tilde{u}_1 = 0$ satisfying

$$(3.16) \qquad \tilde{u}_1(n) = u_1(n)(1 + o(1))$$

we are done.

To show this, we begin with

$$(3.17) \qquad \tau_1 \tilde{u}_1(n) = \Delta(n)\tilde{u}_1(n), \qquad \Delta(n) = b_1(n) - \tilde{b}_1(n),$$

and use the solution formula for the inhomogeneous equation ([15], eqn. (1.48)) to obtain the following equation

$$(3.18) \qquad \tilde{u}_1(n) = u_1(n) - \sum_{j=n+1}^{n_0} u_1(n)u_1(j)(Q_1(n) - Q_1(j))\Delta(j)\tilde{u}_1(j),$$

where Q_1 is defined as Q_0 but with u_1 in place of u_0. Formally, letting $n_0 \to \infty$, and setting

$$(3.19) \qquad \tilde{u}_1(n) = u_1(n)v(n)$$

we obtain

$$(3.20) \qquad v(n) = 1 - \sum_{j=n+1}^{\infty} u_1(j)^2(Q_1(n) - Q_1(j))\Delta(j)v(j).$$

If we can show existence of a solution $v(n) = 1 + o(1)$ of this last equation, we are done. For this it suffices to verify the assumptions of [15], Lemma 7.8. Hence we need to estimate the kernel of the above sum equation. Using

$$|Q_1(n) - Q_1(j)| \leq \sum_{k=n}^{j} \frac{1}{a(k)u_0(k)u_0(k+1)\sqrt{Q_0(k)Q_0(k+1)}}$$

$$(3.21) \qquad \leq \frac{1}{a_0} \sum_{k=n}^{j} \frac{1}{u_0(k)^2 Q_0(k)}$$

and

$$(3.22) \qquad |\Delta(j)| \leq \frac{const}{u_0(j)^6 Q_0(j)^3}$$

we obtain by Lemma 3.1

$$|u_1(j)^2 (Q_1(n) - Q_1(j))\Delta(j)| \leq \frac{const}{u_0(j)^4 Q_0(j)^2} \sum_{k=n}^{j} \frac{1}{u_0(k)^2 Q_0(k)}$$

$$(3.23) \qquad \leq const \frac{\ln(j)}{j^2}.$$

Thus we can apply [15], Lemma 7.8 to conclude existence of a solution of type (3.16) which finishes the proof of Theorem 2.1.

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