Reconstructing Jacobi Matrices from Three Spectra

Johanna Michor and Gerald Teschl

Abstract. Cut a Jacobi matrix into two pieces by removing the n-th column and n-th row. We give necessary and sufficient conditions for the spectra of the original matrix plus the spectra of the two submatrices to uniquely determine the original matrix. Our result contains Hochstadt's theorem as a special case.

1. Introduction

The topic of this paper is inverse spectral theory for Jacobi matrices, that is, matrices of the form

$$H = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-2} & b_{N-1} & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix} \tag{1.1}$$

This is an old problem closely related to the moment problem (see [9] and the references therein), which has attracted considerable interest recently (see, e.g., [1] and the references therein, [3], [4], [8]). For analogous results in the case of Sturm-Liouville operators see [2], [6], and [7]. In this note we want to investigate the following question: Remove the n-th row and the n-th column from H and denote the resulting submatrices by H_- (from b_1 to b_{n-1}) respectively H_+ (from b_{n+1} to b_N). When do the spectra of these three matrices determine the original matrix H? We will show that this is the case if and only if H_- and H_+ have no eigenvalues in common.

From a physical point of view such a model describes a chain of N particles coupled via springs and fixed at both end points (see [11], Section 1.5). Determining the eigenfrequencies of this system and the one obtained by keeping one particle fixed, one can uniquely reconstruct the masses and spring constants. Moreover, these results can be applied to completely integrable systems, in particular the Toda lattice (see e.g., [11]).

1991 *Mathematics Subject Classification.* Primary 36A10, 39A70; Secondary 34B24, 34L05.
Key words and phrases. Jacobi matrices, spectral theory, trace formulas, Hochstadt's theorem.

2. Main result

To set the stage let us introduce some further notation. We denote the spectra of the matrices introduced in the previous section by

$$\sigma(H) = \{\lambda_j\}_{j=1}^{N}, \quad \sigma(H_-) = \{\mu_k^-\}_{k=1}^{n-1}, \quad \sigma(H_+) = \{\mu_l^+\}_{l=1}^{N-n}. \tag{2.1}$$

Moreover, we denote by $(\mu_j)_{j=1}^{N-1}$ the ordered eigenvalues of H_- and H_+ (listing common eigenvalues twice) and recall the well-known formula (see [1], Theorem 2.4 and Theorem 2.8)

$$g(z,n) = -\frac{\prod_{j=1}^{N-1}(z-\mu_j)}{\prod_{j=1}^{N}(z-\lambda_j)} = \frac{-1}{z - b_n + a_n^2 m_+(z,n) + a_{n-1}^2 m_-(z,n)}, \tag{2.2}$$

where $g(z,n)$ are the diagonal entries of the resolvent $(H-z)^{-1}$ and $m_\pm(z,n)$ are the Weyl m-functions corresponding to H_- and H_+. The Weyl functions $m_\pm(z,n)$ are Herglotz and hence have a representation of the following form

$$m_-(z,n) = \sum_{k=1}^{n-1} \frac{\alpha_k^-}{\mu_k^- - z}, \qquad \alpha_k^- > 0, \quad \sum_{k=1}^{n-1} \alpha_k^- = 1, \tag{2.3}$$

$$m_+(z,n) = \sum_{l=1}^{N-n} \frac{\alpha_l^+}{\mu_l^+ - z}, \qquad \alpha_l^+ > 0, \quad \sum_{l=1}^{N-n} \alpha_l^+ = 1. \tag{2.4}$$

With this notation our main result reads as follows

Theorem 2.1. *To each Jacobi matrix H we can associate spectral data*

$$\{\lambda_j\}_{j=1}^{N}, \quad (\mu_j, \sigma_j)_{j=1}^{N-1}, \tag{2.5}$$

where $\sigma_j = +1$ if $\mu_j \in \sigma(H_+)\backslash\sigma(H_-)$, $\sigma_j = -1$ if $\mu_j \in \sigma(H_-)\backslash\sigma(H_+)$, and

$$\sigma_j = \frac{a_n^2 \alpha_l^+ - a_{n-1}^2 \alpha_k^-}{a_n^2 \alpha_l^+ + a_{n-1}^2 \alpha_k^-} \tag{2.6}$$

if $\mu_j = \mu_k^- = \mu_l^+$.

Then these spectral data satisfy

(i) $\lambda_1 < \mu_1 \le \lambda_2 \le \mu_2 \le \cdots < \lambda_N$,
(ii) $\sigma_j = \sigma_{j+1} \in (-1,1)$ *if* $\mu_j = \mu_{j+1}$ *and* $\sigma_j \in \{\pm 1\}$ *if* $\mu_j \ne \mu_i$ *for* $i \ne j$

and uniquely determine H. Conversely, for every given set of spectral data satisfying (i) and (ii), there is a corresponding Jacobi matrix H.

Proof. We first consider the case where H_- and H_+ have no eigenvalues in common. The interlacing property (i) is equivalent to the Herglotz property of $g(z,n)$.

Furthermore, the residues α_i^- can be computed from (2.2),

$$\frac{\prod_{j=1}^{N}(z-\lambda_j)}{\prod_{k=1}^{n-1}(z-\mu_k^-)\prod_{l=1}^{N-n}(z-\mu_l^+)} = z - b_n + a_n^2\sum_{l=1}^{N-n}\frac{\alpha_l^+}{z-\mu_l^+} + a_{n-1}^2\sum_{k=1}^{n-1}\frac{\alpha_k^-}{z-\mu_k^-}, \tag{2.7}$$

and are given by $\alpha_i^- = a_{n-1}^{-2}\beta_i^-$, where

$$\beta_i^- = -\frac{\prod_{j=1}^{N}(\mu_i^- - \lambda_j)}{\prod_{l\neq i}(\mu_i^- - \mu_l^-)\prod_{l=1}^{N-n}(\mu_i^- - \mu_l^+)}, \qquad a_{n-1}^2 = \sum_{i=1}^{n-1}\beta_i^-. \tag{2.8}$$

Similarly, $\alpha_l^+ = a_n^{-2}\beta_l^+$, where

$$\beta_l^+ = -\frac{\prod_{j=1}^{N}(\mu_l^+ - \lambda_j)}{\prod_{k=1}^{n-1}(\mu_l^+ - \mu_k^-)\prod_{p\neq l}(\mu_l^+ - \mu_p^+)}, \qquad a_n^2 = \sum_{l=1}^{N-n}\beta_l^+. \tag{2.9}$$

Hence $m_\pm(z,n)$ are uniquely determined and thus $H_\pm$ by standard results from the moment problem. The only remaining coefficient b_n follows from the well-known trace formula

$$b_n = \mathrm{tr}(H) - \mathrm{tr}(H_-) - \mathrm{tr}(H_+) = \sum_{j=1}^{N}\lambda_j - \sum_{k=1}^{n-1}\mu_k^- - \sum_{l=1}^{N-n}\mu_l^+. \tag{2.10}$$

Conversely, suppose we have the spectral data given. Then we can define a_n, a_{n-1}, b_n, α_k^-, α_l^+ as above. By (i), α_k^- and α_l^+ are positive and hence give rise to $H_\pm$. Together with a_n, a_{n-1}, b_n we have thus defined a Jacobi matrix H. By construction, the eigenvalues μ_k^-, μ_l^+ are the right ones and also (2.2) holds for H. Thus λ_j are the eigenvalues of H, since they are the poles of $g(z,n)$.

Next we come to the general case where $\mu_{j_0} = \mu_{k_0}^- = \mu_{l_0}^+$ $(=\lambda_{j_0})$ at least for one j_0. Now some factors in the left hand side of (2.7) will cancel and we can no longer compute $\beta_{k_0}^-$, $\beta_{l_0}^+$, but only $\gamma_{j_0} = \beta_{k_0}^- + \beta_{l_0}^+$. However, by definition of σ_{j_0} we have

$$\beta_{k_0}^- = \frac{1-\sigma_{j_0}}{2}\gamma_{j_0}, \qquad \beta_{l_0}^+ = \frac{1+\sigma_{j_0}}{2}\gamma_{j_0}. \tag{2.11}$$

Now we can proceed as before to see that H is uniquely determined by the spectral data.

Conversely, we can also construct a matrix H from given spectral data, but it is no longer clear that λ_j is an eigenvalue of H unless it is a pole of $g(z,n)$. However, in the case $\lambda_{j_0} = \mu_{k_0}^- = \mu_{l_0}^+$ we can glue the eigenvectors u_- of H_- and u_+ of H_+ to give an eigenvector $(u_-, 0, u_+)$ corresponding to λ_{j_0} of H. □

The special case where we remove the first row and the first column (in which case H_- is not present) corresponds to Hochstadt's theorem [5]. Similar results for (quasi-)periodic Jacobi operators can be found in [10].

Acknowledgments. We thank the referee for useful suggestions with respect to the literature.

References

[1] F. Gesztesy and B. Simon, *m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices*, J. Anal. Math. **73**, 267–297 (1997).

[2] F. Gesztesy and B. Simon, On the determination of a potential from three spectra, in *Differential operators and spectral theory*, 85–92, *Amer. Math. Soc. Transl. Ser. 2*, **189**, Amer. Math. Soc., Providence, RI, 1999.

[3] G. M. L. Gladwell, *On isospectral spring-mass systems*, Inverse Probl. **11**, 591–602 (1995).

[4] P. C. Gibson, *Inverse spectral theory of finite Jacobi matrices*, Trans of the Amer. Math. Soc. **354**, 4703–4749 (2002).

[5] H. Hochstadt, *On the construction of a Jacobi matrix from spectral data*, Lin. Algebra Appl. **8**, 435–446, 1974.

[6] R. Hryniv and Ya. Mykytyuk, *Inverse spectral problems for Sturm-Liouville operators with singular potentials. Part III: Reconstruction by three spectra*, J. Math. Anal. Appl. **248**, 626–646, (2003)

[7] V. Pivovarchik, *An inverse Sturm-Liouville problem by three spectra*, Integral Equations Operator Theory **34** no. 2, 234–243, (1999).

[8] C.-T. Shieh, *Reconstruction of a Jacobi matrix with mixed data*, preprint.

[9] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Advances in Math. **137**, 82–203 (1998).

[10] G. Teschl, *Trace Formulas and Inverse Spectral Theory for Jacobi Operators*, Comm. Math. Phys. **196**, 175–202 (1998).

[11] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

Institut für Mathematik, Strudlhofgasse 4, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: `Johanna.Michor@esi.ac.at`

Institut für Mathematik, Strudlhofgasse 4, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: `Gerald.Teschl@univie.ac.at`
URL: `http://www.mat.univie.ac.at/~gerald/`