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DEFORMING THE POINT SPECTRA OF ONE-DIMENSIONAL DIRAC OPERATORS

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Abstract. We provide a method of inserting and removing any finite number of prescribed eigenvalues into spectral gaps of a given one-dimensional Dirac operator. This is done in such a way that the original and deformed operator are unitarily equivalent when restricted to the complement of the subspace spanned by the newly inserted eigenvalue. Moreover, the unitary transformation operator which links the original operator to its deformed version is explicitly determined.

1. Introduction

Methods of inserting (and removing) eigenvalues in spectral gaps of a given one-dimensional Schrödinger operator H have been quite popular recently. This is due to their important role in diverse fields such as the inverse scattering approach introduced by Deift and Trubowitz [3], level comparison theorems (cf. [1] and the literature cited therein), and as a tool for constructing soliton solutions of the Korteweg–de Vries hierarchy relative to known background solutions (see, e.g., [7], and the references therein). For more information and a brief historic account we refer the reader to [6], [8].

It is surprising that even though Dirac operators are as important in applications as Sturm–Liouville operators, no analogous methods are available for these operators (except for the case of supersymmetric Dirac operators where results from Sturm–Liouville operators apply). This lack is clearly connected to the fact that Dirac operators are not bounded from below and hence cannot be factored into a product of type A^*A (which would be necessarily non-negative). However, this factorization lies at the heart of methods for inserting eigenvalues into the spectra of Sturm–Liouville operators (cf. [2]). This shows that, for inserting eigenvalues into the spectra of Dirac operators, an entirely new strategy is needed. Our new approach is modeled after "Hilbert's hotel". That is, our idea is to use a transformation operator which "compresses" the underlying Hilbert space a little such that the range has codimension one. This way we create a one-dimensional subspace to accommodate the new eigenvalue. On the remainder of the Hilbert space we require the transform to be unitary such that all other spectral features of the original operator are preserved.

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Clearly, not any transformation can be used since, in general, the transformed operator will not be a Dirac operator. However, a generalized version of a transformation found in [8] will do the trick.

Let $I = (a, b) \subseteq \mathbb{R}$ (with $-\infty \le a < b \le \infty$) be an arbitrary interval, $m \in \mathbb{R}^+ = [0, \infty)$, and $\phi_{\rm am}, \phi_{\rm el}, \phi_{\rm sc} \in L^1_{loc}(I, \mathbb{R})$ real-valued. Consider the Dirac differential expression

$$\tau = \sigma_2 \frac{1}{\mathrm{i}} \frac{d}{dx} + \phi(x). \tag{1.1}$$

Here

$$\phi(x) = \phi_{\rm el}(x)\mathbb{1} + \phi_{\rm am}(x)\sigma_1 + (m + \phi_{\rm sc}(x))\sigma_3, \tag{1.2}$$

where σ_1, σ_2, σ_3 denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.3}$$

and m, $\phi_{\rm sc}$, $\phi_{\rm el}$, and $\phi_{\rm am}$ are interpreted as mass, scalar potential, electrostatic potential, and anomalous magnetic moment, respectively (see [12], Chapter 4). We don't include a magnetic moment since it can be easily eliminated by a simple gauge transformation (there is also a gauge transformation which gets rid of $\phi_{\rm am}$; see [10], Section 7.1.1).

If τ is limit point (*l.p.*) at both a and b (cf., e.g., [10], [13], [14]), then τ gives rise to a unique self-adjoint operator H when defined maximally. Otherwise, we fix a boundary condition at each endpoint where τ is limit circle (*l.c.*).

By $u_+(z, x)$ (resp. $u_-(z, x)$) we will denote (non identically vanishing) solutions of the differential equation $\tau u = z u$, $z \in \mathbb{C}$, which are integrable near b (resp. a) and fulfill the boundary condition of H at b (resp. a) if any (i.e., if τ is limit circle at b (resp. a)). A sufficient criterion for $u_\pm(z, x)$ to exist is $z \in \mathbb{C}\backslash\sigma_{ess}(H_{c,\pm})$ or $z \in \sigma_p(H_{c,\pm})$, where $\sigma_p(.)$, $\sigma_{ess}(.)$ denotes the point, essential spectrum, respectively. Here $H_{c,-}$ (resp. $H_{c,+}$), $c \in I$ denotes the self-adjoint operators associated with τ on $L^2((a, c), \mathbb{C}^2)$ (resp. $L^2((c, b), \mathbb{C}^2)$) obtained from H by imposing the additional boundary condition $f_1(c) = 0$. Then $H_{c,-} \oplus H_{c,+}$ is a rank one resolvent perturbation of H and hence $\sigma_{ess}(H) = \sigma_{ess}(H_{c,-}) \cup \sigma_{ess}(H_{c,+})$ (cf. [14], Korollar 6.2).

Using this notation, the operator H is explicitly given by

$$H : \begin{array}{ccc} \mathfrak{D}(H) & \to & L^2(I, \mathbb{C}^2) \\ f & \mapsto & \tau f \end{array}, \tag{1.4}$$

where

$$\mathfrak{D}(H) = \{ f \in L^2(I, \mathbb{C}^2) | \begin{array}{l} f \in AC_{loc}(I, \mathbb{C}^2),\ \tau f \in L^2(I, \mathbb{C}^2), \\ W_a(u_-(z_0), f) = W_b(u_+(z_0), f) = 0 \} \end{array} \tag{1.5}$$

with

$$W_x(f, g) = f_1(x) g_2(x) - f_2(x) g_1(x) \tag{1.6}$$

the usual Wronskian (we remark that the the limit $W_{a,b}(.,..) = \lim_{x\to a,b} W_x(.,..)$ exists for functions as in (1.5)).

2. Construction of a transformation operator

Fix $n \in \mathbb{N}$ and let k be a positive definite n by n matrix with coefficients in $L^1_{loc}(I)$. We pick $\mathfrak{H} = L^2(I, \mathbb{C}^n; k\,dx)$ to be the underlying Hilbert space. The scalar product and norm in $\mathfrak{H}$ are given by

$$\langle f, g\rangle = \int_a^b \overline{f(t)} k(t) g(t) dt, \quad \|f\|^2 = \langle f, f\rangle. \tag{2.1}$$

Denote by $\mathfrak{H}_-$ (resp. $\mathfrak{H}_+$) functions in $L^2_{loc}(I, \mathbb{C}^n; k\,dx)$ which are in $\mathfrak{H}$ near a (resp. b) and choose a function $u \in \mathfrak{H}_-$ plus a constant $\gamma \in [-\|u\|^{-2}, \infty) \cup \{\infty\}$. Define

$$c_\gamma(x) = \frac{1}{\gamma} + \langle u, u\rangle_a^x, \quad \gamma \neq 0 \tag{2.2}$$

(setting $\infty^{-1} = 0$), where

$$\langle f, g\rangle_y^x = \int_y^x \overline{f(t)} k(t) g(t) dt. \tag{2.3}$$

Consider the following linear transformation

$$\begin{array}{rccl} U_\gamma : & \mathfrak{H} & \to & L^2_{loc}(I, \mathbb{C}^n, k\,dx) \\ & f(x) & \mapsto & f(x) - u_\gamma(x)\langle u, f\rangle_a^x \end{array}, \tag{2.4}$$

($U_0 = \mathbb{1}$), where

$$u_\gamma(x) = \frac{u(x)}{c_\gamma(x)}, \tag{2.5}$$

($u_0 = 0$). We note that U_γ can be defined on $\mathfrak{H}_-$ and $U_\gamma u = \gamma^{-1} u_\gamma$, $\gamma \neq 0$. Furthermore,

$$\overline{u_\gamma(x)} k(x) u_\gamma(x) = -\frac{d}{dx}\frac{1}{c_\gamma(x)}, \tag{2.6}$$

and hence

$$\|u_\gamma\|^2 = \begin{cases} \gamma, & u \notin \mathfrak{H} \\ \frac{\gamma^2\|u\|^2}{1+\gamma\|u\|^2}, & u \in \mathfrak{H} \end{cases} \tag{2.7}$$

implying $u_\gamma \in \mathfrak{H}$ if $-\|u\|^{-2} < \gamma < \infty$. If $\gamma = -\|u\|^{-2}$, $\gamma = \infty$ we only have that u_γ is in $\mathfrak{H}_-$, $\mathfrak{H}_+$, respectively. In addition, we remark that for $f_\gamma = U_\gamma f$ we have

$$\overline{u_\gamma(x)} k(x) f_\gamma(x) = \frac{d}{dx}\frac{\langle u, f\rangle_a^x}{c_\gamma(x)}, \tag{2.8}$$

$$\overline{f_\gamma(x)} k(x) f_\gamma(x) = \overline{f(x)} k(x) f(x) - \frac{d}{dx}\frac{|\langle u, f\rangle_a^x|^2}{c_\gamma(x)}. \tag{2.9}$$

Integrating over x and taking limits (if $\gamma = \infty$ use Cauchy-Schwarz) shows

$$\langle u_\gamma, f_\gamma\rangle_a^x = \begin{cases} c_\gamma(x)^{-1}\langle u, f\rangle_a^x, & \gamma \in \mathbb{R} \\ \frac{\langle u, f\rangle}{\|u\|^2} - c_\infty(x)^{-1}\langle u, f\rangle_x^b, & \gamma = \infty \end{cases}, \tag{2.10}$$

$$\langle f_\gamma, f_\gamma\rangle_a^x = \langle f, f\rangle_a^x - \frac{|\langle u, f\rangle_a^x|^2}{c_\gamma(x)}. \tag{2.11}$$

Clearly, the last equation implies $U_\gamma : \mathfrak{H} \to \mathfrak{H}$. In addition, we remark that this also shows $U_\gamma : \mathfrak{H}_- \to \mathfrak{H}_-$.

Denote by P, P_γ the orthogonal projections onto the one-dimensional subspaces of $\mathfrak{H}$ spanned by u, u_γ (set $P, P_\gamma = 0$ if $u, u_\gamma \notin \mathfrak{H}$), respectively. Define

$$
\begin{array}{rccl}
U_\gamma^{-1}: & \mathfrak{H} & \to & L^2_{loc}(I, \mathbb{C}^n, k\,dx) \\
 & g(x) & \mapsto & \left\{ \begin{array}{ll} g(x) + u(x)\langle u_\gamma, g\rangle_a^x, & \gamma \in \mathbb{R} \\ g(x) - u(x)\langle u_\infty, g\rangle_x^b, & \gamma = \infty \end{array} \right.
\end{array}
\tag{2.12}
$$

and note

$$
c_\gamma^{-1}(x) = \left\{ \begin{array}{cc} \gamma - \langle u_\gamma, u_\gamma\rangle_a^x, & \gamma \in \mathbb{R} \\ \|u\|^{-2} + \langle u_\infty, u_\infty\rangle_x^b, & \gamma = \infty \end{array} \right. . \tag{2.13}
$$

As before one can show $U_\gamma^{-1} : (\mathbb{1} - P_\gamma)\mathfrak{H} \to \mathfrak{H}$ and one verifies

$$
\begin{array}{ll}
U_\gamma U_\gamma^{-1} = \mathbb{1},\ U_\gamma^{-1} U_\gamma = \mathbb{1}, & \gamma \in \mathbb{R}, \\
U_\infty U_\infty^{-1} = \mathbb{1},\ U_\infty^{-1} U_\infty = \mathbb{1} - P, & \gamma = \infty,
\end{array}
\tag{2.14}
$$

If $P = 0$, $\gamma \in (-\|u\|^{-2}, \infty)$, then $U_\gamma U_\gamma^{-1} = \mathbb{1}$ should be replaced by $U_\gamma U_\gamma^{-1} = \mathbb{1}_{(\mathbf{1} - P_\gamma)\mathfrak{H}}$ since $U_\gamma^{-1} u_\gamma \notin \mathfrak{H}$ by

$$
U_\gamma^{-1} u_\gamma = \left\{ \begin{array}{cc} \gamma u, & \gamma \in \mathbb{R} \\ \|u\|^{-2} u, & \gamma = \infty \end{array} \right. . \tag{2.15}
$$

Summarizing,

Lemma 2.1. *The operator U_γ is unitary from $(\mathbb{1} - P)\mathfrak{H}$ onto $(\mathbb{1} - P_\gamma)\mathfrak{H}$ with inverse U_γ^{-1}. If $P, P_\gamma \neq 0$, then U_γ can be extended to a unitary transformation $\tilde{U}_\gamma$ on $\mathfrak{H}$ by*

$$
\tilde{U}_\gamma = U_\gamma(\mathbb{1} - P_\gamma) + \sqrt{1 + \gamma\|u\|^2}\, U_\gamma P_\gamma. \tag{2.16}
$$

Proof. Equation (2.10) shows that U_γ maps $(\mathbb{1} - P)\mathfrak{H}$ onto $(\mathbb{1} - P_\gamma)\mathfrak{H}$. Unitarity follows from (2.11) and

$$
\lim_{x \to b} \frac{|\langle u, f\rangle_a^x|^2}{\langle u, u\rangle_a^x} = 0 \tag{2.17}
$$

for any $f \in \mathfrak{H}$ if $u \notin \mathfrak{H}$. In fact, suppose $\|f\| = 1$, pick y and $x > y$ so large that $\langle f, f\rangle_y^b \leq \varepsilon/2$ and $\langle u, u\rangle_a^y / \langle u, u\rangle_a^x \leq \varepsilon/2$. Splitting up the sum in the numerator and applying Cauchy's inequality then shows that the limit of (2.17) is smaller than ε. □

We remark that (2.11) plus the polarization identity implies

$$
\langle f_\gamma, g_\gamma\rangle_a^x = \langle f, g\rangle_a^x - \frac{\langle f, u\rangle_a^x \langle u, g\rangle_a^x}{c_\gamma(x)}, \tag{2.18}
$$

where $f_\gamma = U_\gamma f$, $g_\gamma = U_\gamma g$.

3. Inserting a single eigenvalue

Now we turn to the Dirac operator H defined in the Introduction. We will choose (λ_1, γ_1) satisfying

Hypothesis H.3.1. *Suppose $(\lambda, \gamma) \in \mathbb{R}^2$ satisfies the following conditions.*
(i). $u_-(\lambda, x)$ exists.
(ii). $\gamma \in [-\|u_-(\lambda)\|^{-2}, \infty) \cup \{\infty\}$.
(iii). If $u_-(\lambda) \in \mathfrak{H}$, then $\lambda \in \sigma_p(H)$.

and use Lemma 2.1 with $u = u_-(\lambda_1)$, $\gamma = \gamma_1$ to prove

Theorem 3.2. *Suppose (H.3.1) and let H_{γ_1} be the operator associated with*

$$H_{\gamma_1} f = \tau_{\gamma_1} f, \quad \mathfrak{D}(H_{\gamma_1}) = \{\, f \in \mathfrak{H} | \, f \in AC_{loc}(I, \mathbb{C}^2); \tau_{\gamma_1} f \in \mathfrak{H}; \; W_a(u_{\gamma_1,-}(\lambda_1), f) = W_b(u_{\gamma_1,-}(\lambda_1), f) = 0\}, \tag{3.1}$$

where

$$\phi_{\gamma_1} = \phi + \frac{u_-(\lambda_1) \otimes_\sigma u_-(\lambda_1)}{c_{\gamma_1}(\lambda_1, x)}, \quad f \otimes_\sigma g = \frac{f \otimes (\sigma_2 g) + (\sigma_2 f) \otimes g}{\mathrm{i}}, \tag{3.2}$$

and

$$u_{\gamma_1,-}(\lambda_1, x) = \frac{u_-(\lambda_1, x)}{c_{\gamma_1}(\lambda_1, x)}, \quad c_{\gamma_1}(\lambda_1, x) = \frac{1}{\gamma_1} + \langle u_-(\lambda_1), u_-(\lambda_1)\rangle_a^x. \tag{3.3}$$

Then

$$H_{\gamma_1}(\mathbb{1} - P_{\gamma_1}(\lambda_1)) = U_{\gamma_1} H U_{\gamma_1}^{-1}(\mathbb{1} - P_{\gamma_1}(\lambda_1)) \tag{3.4}$$

and $\tau_{\gamma_1} u_{\gamma_1,-}(\lambda_1) = \lambda_1 u_{\gamma_1,-}(\lambda_1)$.

Proof. Only the case $\gamma \neq 0$ is of interest. The claim $\tau_{\gamma_1} u_{\gamma_1,-}(\lambda_1) = \lambda_1 u_{\gamma_1,-}(\lambda_1)$ is straightforward and implies that τ_{γ_1} is *l.p.* at a, b if $\gamma = \infty, -\|u_-(\lambda_1)\|^{-2}$, respectively. Moreover, let $f \in \mathfrak{D}(H)$ then another straightforward calculation shows

$$\tau_{\gamma_1}(U_{\gamma_1} f) = U_{\gamma_1}(\tau f) \tag{3.5}$$

and it remains to compute $U_{\gamma_1}\mathfrak{D}(H)$. It suffices to vindicate

$$(\mathbb{1} - P_{\gamma_1}(\lambda_1)) U_{\gamma_1} \mathfrak{D}(H) \subseteq (\mathbb{1} - P_{\gamma_1}(\lambda_1)) \mathfrak{D}(H_{\gamma_1}) \tag{3.6}$$

since $(\mathbb{1} - P_{\gamma_1}(\lambda_1)) U_{\gamma_1} \mathfrak{D}(H)$ cannot be properly contained in $(\mathbb{1} - P_{\gamma_1}(\lambda_1)) \mathfrak{D}(H_{\gamma_1})$ by the property of self-adjoint operators being maximal. Only the boundary conditions are not obvious. If $\gamma \in \mathbb{R}$ the formula

$$W_x(u_{\gamma_1,-}(\lambda_1), U_{\gamma_1} f) = \frac{W_x(u_-(\lambda_1), f)}{c_{\gamma_1}(\lambda_1, x)} \tag{3.7}$$

reveals $W_a(u_{\gamma_1,-}(\lambda_1), U_{\gamma_1} f) = 0$ for $f \in \mathfrak{D}(H)$ (if $\gamma = \infty$, then τ_{γ_1} is *l.p.* at a). For the boundary condition at b we can assume $\gamma \neq -\|u_-(\lambda_1)\|^{-2}$. If $u_-(\lambda_1) \in \mathfrak{H}$, then (3.7) shows $W_b(u_{\gamma_1,-}(\lambda_1), U_{\gamma_1} f) = 0$ for $f \in \mathfrak{D}(H)$. Otherwise, that is, if $u_-(\lambda_1) \notin \mathfrak{H}$ we use

$$|W_x(u_{\gamma_1,-}(\lambda_1), U_{\gamma_1} f)|^2 = \frac{|\langle u_-(\lambda_1), (\tau - \lambda_1) f\rangle_a^x|^2}{c_{\gamma_1}(\lambda_1, x)^2}, \tag{3.8}$$

which tends to zero as $x \to b$ for $f \in \mathfrak{D}(H)$ by (2.17). □

We remark that explicitly (3.2) reads

$$\phi_{\gamma_1,\mathrm{el}}(x) = \phi_{\mathrm{el}}(x), \tag{3.9}$$

$$\phi_{\gamma_1,\mathrm{am}}(x) = \phi_{\mathrm{am}}(x) + \frac{u_{-,1}(\lambda_1, x)^2 - u_{-,2}(\lambda_1, x)^2}{c_{\gamma_1}(\lambda_1, x)}, \tag{3.10}$$

$$\phi_{\gamma_1,\mathrm{sc}}(x) = \phi_{\mathrm{sc}}(x) - 2\frac{u_{-,1}(\lambda_1, x) u_{-,2}(\lambda_1, x)}{c_{\gamma_1}(\lambda_1, x)}. \tag{3.11}$$

Corollary 3.3. *Suppose $u_-(\lambda_1) \notin \mathfrak{H}$.*

(i). If $\gamma_1 > 0$ then H and $(\mathbb{1} - P_{\gamma_1}(\lambda_1))H_{\gamma_1}$ are unitarily equivalent. Moreover, H_{γ_1} has the additional eigenvalue λ_1 with eigenfunction $u_{\gamma_1,-}(\lambda_1)$.

(ii). If $\gamma_1 = \infty$ then H and H_{γ_1} are unitarily equivalent.

Suppose $u_-(\lambda_1) \in \mathfrak{H}$ (i.e., λ_1 is an eigenvalue of H).

(i). If $\gamma_1 \in (-\|u_-(\lambda_1)\|^{-2}, \infty)$ than H and H_{γ_1} are unitarily equivalent (using $\tilde{U}_{\gamma_1}$).

(ii). If $\gamma_1 = -\|u_-(\lambda_1)\|^{-2}, \infty$ then $(\mathbb{1} - P(\lambda_1))H$, H_{γ_1} are unitarily equivalent, that is, the eigenvalue λ_1 is removed.

The following can be verified directly.

Lemma 3.4. *Let $u \in AC_{loc}(I, \mathbb{C}^2)$ fulfill $\tau u = zu$ (with $z \in \mathbb{C}\backslash\{\lambda_1\}$) and let*

$$v(z,x) = u(z,x) + \frac{u_{\gamma_1,-}(\lambda_1,x)}{z-\lambda_1} W_x(u_-(\lambda_1), u(z)). \tag{3.12}$$

Then $v \in AC_{loc}(I, \mathbb{C}^2)$ and v fulfills $\tau_{\gamma_1} v = zv$. If u is square integrable near a and fulfills the boundary condition at a (if any) we have $v = U_{\gamma_1} u$. We also note

$$|v(z,x)|^2 = |u(z,x)|^2 - \frac{1}{|z-\lambda_1|^2} \frac{d}{dx} \left(\frac{|W_x(u_-(\lambda_1), u(z))|^2}{c_{\gamma_1}(\lambda_1,x)} \right), \tag{3.13}$$

and if $\hat{u}$, $\hat{v}$ are constructed analogously then

$$\begin{aligned} W_x(v(z), \hat{v}(\hat{z})) \;&=\; W_x(u(z), \hat{u}(\hat{z})) - \frac{1}{c_{\gamma_1}(\lambda_1,x)} \times \\ & \frac{z-\hat{z}}{(z-\lambda_1)(\hat{z}-\lambda_1)} W_x(u_-(\lambda_1), u(z)) W_x(u_-(\lambda_1), \hat{u}(\hat{z})). \end{aligned} \tag{3.14}$$

In addition, the solutions

$$u_{\pm,\gamma_1}(z,x) = u_\pm(z,x) + \frac{u_{\gamma_1,-}(\lambda_1,x)}{z-\lambda_1} W_x(u_-(\lambda_1), u_\pm(z)). \tag{3.15}$$

are square integrable near a, b and satisfy the boundary condition of H_{γ_1} at a, b, respectively.

Remark 3.5. *From (3.15) one can easily compute the Weyl m-functions corresponding to H. Proceeding as in* [6], [9] *one can then obtain an alternate proof for Corollary 3.3.*

Since we have already seen, that our method does not preserve *l.p./l.c.* properties we want to discuss conditions for τ_{γ_1} to be *l.p.* at a, b. Let $c \in I$ and let $H_{+,c}$ denote a self-adjoint operator associated with τ on (c, b) and the boundary condition induced by $u_-(\lambda_1)$ at c (i.e., $W_c(f, u_-(\lambda_1)) = 0$, $f \in \mathfrak{D}(H_{c,+})$).

Hypothesis H.3.6. *Suppose one of the following spectral conditions (i)–(ii) holds.*

(i). $\sigma_{ess}(H_{+,c}) \neq \emptyset$.

(ii). $\sigma(H_{+,c}) = \sigma_d(H_{+,c}) = \{\lambda_n\}_{n\in\mathbb{Z}}$ with $\sum_{n\in\mathbb{Z}} (1+\lambda_n^2)^{-1} = \infty$.

All conditions (i)–(ii) imply that τ is *l.p.* at b (since τ *l.c.* at b implies that the resolvent of $H_{+,c}$ is Hilbert–Schmidt) and we easily obtain (using (3.13))

Theorem 3.7. *(i). If $\gamma_1 \neq \infty$, then τ_{γ_1} is l.p. at a if and only if τ is. Otherwise, that is, if $\gamma = \infty$, then τ_{γ_1} is l.p. at a.*

(ii). If $\gamma_1 \neq -\|u_-(\lambda_1)\|^{-2}$, then τ_{γ_1} is l.c. at b if τ is. If $\gamma_1 = -\|u_-(\lambda_1)\|^{-2} (\neq 0)$, then τ_{γ_1} is l.p. at b.
(iii). Assume (H.3.6), then τ_{γ_1} is l.p. at b.

Remark 3.8. *(i). Removing an eigenvalue from an operator which is l.c. at b yields an operator which is l.p.. This shows that one cannot insert additional eigenvalues into an operator which is l.c. at b (remove this eigenvalue again to obtain a contradiction).*
(ii). Clearly we can interchange the role of a and b. One only has to interchange a, b in the text and $\langle .,..\rangle_a^x$, $\langle .,..\rangle_x^b$ in the formulas.
(iii). As long as $u_-(\lambda_1)$ exists (e.g., τ is l.c. at a) our method can be used to insert additional eigenvalues into the spectrum of H (cf. [11]*).*
(iv). It is well-known that methods of inserting eigenvalues into the spectra of one-dimensional Schrödinger operators are connected with Bäcklund (Darboux) transformations of the (modified) Korteweg–de Vries hierarchy. This raises the question whether our method is connected with transformations of the AKNS hierarchy. We recall that the AKNS hierarchy is associated with the differential expression

$$\hat{\tau} = -\sigma_3 \frac{1}{\mathrm{i}} \frac{d}{dx} + \frac{1}{2}\sigma_2(p(x) + q(x)) + \frac{\mathrm{i}}{2}\sigma_1(p(x) - q(x)). \tag{3.16}$$

Transforming $\hat{\tau}$ to our representation yields

$$\tau = U\hat{\tau}U^{-1} = \sigma_2 \frac{1}{\mathrm{i}} \frac{d}{dx} - \frac{1}{2}\sigma_3(p(x) + q(x)) + \frac{\mathrm{i}}{2}\sigma_1(p(x) - q(x)), \tag{3.17}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}. \tag{3.18}$$

Since we are interested in the self-adjoint case we require $p = \overline{q}$. This corresponds to the nonlinear Schrödinger equation. If in addition $\mathrm{Re}(p) = 0$ we have the case of supersymmetric operators which is connected to the modified Korteweg–de Vries hierarchy. Since our transformation does not leave (3.17) invariant, it cannot correspond to a transformation of the AKNS system. This becomes even more evident in the supersymmetric case, since inserting one eigenvalue $\lambda_1 \neq 0$ must necessarily destroy supersymmetry.

4. Inserting finitely many eigenvalues

Finally we demonstrate how to iterate this method. We choose a given background operator H and pick (λ_1, γ_1) according to (H.3.1). Now define the transformation U_{γ_1} and the operator H_{γ_1} as in the previous section. Next we choose (λ_2, γ_2) and define $u_{-,\gamma_1}(\lambda_2) = U_{\gamma_1} u_-(\lambda_2)$ and corresponding operators U_{γ_1,γ_2} and H_{γ_1,γ_2}. Applying this procedure N times leads to

Theorem 4.1. *Let H be the background operator and let $(\lambda_\ell, \gamma_\ell)$, $1 \leq \ell \leq N$ satisfy (H.3.1). Define the following matrices ($1 \leq \ell \leq N$)*

$$C^\ell(x) = \left(\frac{\delta_{r,s}}{\gamma_r} + \langle u_-(\lambda_r), u_-(\lambda_s)\rangle_a^x \right)_{1 \leq r,s \leq \ell}, \tag{4.1}$$

$$C^\ell(f,g)(x) = \begin{pmatrix} C^{\ell-1}(x)_{r,s}, & {\scriptstyle r,s\le \ell-1} \\ \langle f, u_-(\lambda_s)\rangle_a^x, & {\scriptstyle s\le\ell-1, r=\ell} \\ \langle u_-(\lambda_r), g\rangle_a^x, & {\scriptstyle r\le\ell-1, s=\ell} \\ \langle f,g\rangle_a^x, & {\scriptstyle r=s=\ell} \end{pmatrix}_{1\le r,s\le \ell}, \tag{4.2}$$

$$U^\ell(f)(x) = \begin{pmatrix} C^{\ell}(x)_{r,s}, & {\scriptstyle r,s\le \ell} \\ \langle u_-(\lambda_s), f\rangle_a^x, & {\scriptstyle s\le\ell, r=\ell+1} \\ u_-(\lambda_r, x), & {\scriptstyle r\le\ell, s=\ell+1} \\ f, & {\scriptstyle r=s=\ell+1} \end{pmatrix}_{1\le r,s\le \ell+1}. \tag{4.3}$$

Then we have (set $C^0(x)=1$, $U^0(f)=f$)

$$\langle U_{\gamma_1,\dots,\gamma_{\ell-1}}\cdots U_{\gamma_1} f, U_{\gamma_1,\dots,\gamma_{\ell-1}}\cdots U_{\gamma_1} g\rangle_a^x = \frac{\det C^\ell(f,g)(x)}{\det C^{\ell-1}(x)} \tag{4.4}$$

and

$$U_{\gamma_1,\dots,\gamma_\ell}\cdots U_{\gamma_1} f(x) = \frac{\det U^\ell(f)(x)}{\det C^\ell(x)}. \tag{4.5}$$

In particular, we obtain

$$\begin{aligned} c_{\gamma_\ell}(\lambda_\ell, x) &= \frac{1}{\gamma_\ell} + \langle U_{\gamma_1,\dots,\gamma_{\ell-1}}\cdots U_{\gamma_1} u_-(\lambda_\ell), U_{\gamma_1,\dots,\gamma_{\ell-1}}\cdots U_{\gamma_1} u_-(\lambda_\ell)\rangle_a^x \\ &= \frac{\det C^\ell(x)}{\det C^{\ell-1}(x)}. \end{aligned} \tag{4.6}$$

The corresponding operator $H_{\gamma_1,\dots,\gamma_N}$ is associated with

$$\phi_{\gamma_1,\dots,\gamma_N} = \phi + \sum_{\ell=0}^N \frac{\det U^{\ell-1}(u_-(\lambda_\ell)) \otimes_\sigma \det U^{\ell-1}(u_-(\lambda_\ell))}{\det C^\ell(x)}. \tag{4.7}$$

and we have

$$\begin{aligned} &H_{\gamma_1,\dots,\gamma_N}(1\!\!1 - \sum_{j=1}^N P_{\gamma_1,\dots,\gamma_N}(\lambda_j)) \\ &\quad = (U_{\gamma_1,\dots,\gamma_N}\cdots U_{\gamma_1}) H (U_{\gamma_1}^{-1}\cdots U_{\gamma_1,\dots,\gamma_N}^{-1})(1\!\!1 - \sum_{j=1}^N P_{\gamma_1,\dots,\gamma_N}(\lambda_j)), \end{aligned} \tag{4.8}$$

where $P_{\gamma_1,\dots,\gamma_N}(\lambda_j)$ denotes the projection onto the one-dimensional subspace spanned by

$$u_{\gamma_1,\dots,\gamma_N,-}(\lambda_\ell, x) = \gamma_\ell \frac{\det U^\ell(u_-(\lambda_\ell))(x)}{\det C^\ell(x)} \tag{4.9}$$

(the last equation has to be understood as a limit if $\gamma_\ell = \infty$).

Proof. It suffices to prove (4.4), (4.5) which requires a straightforward induction argument using Sylvester's determinant identity ([4], Sect. II.3). The resulting identity

$$\begin{aligned} &\det C^\ell \det C^\ell(f,g) - \det C^\ell(u_-(\lambda_\ell), g)\det C^\ell(f, u_-(\lambda_\ell)) \\ &\qquad = \det C^{\ell-1}\det C^{\ell+1}(f,g), \end{aligned} \tag{4.10}$$

together with (2.18) then proves (4.4). Similarly,

$$\det C^{\ell} \det U^{\ell-1}(f) - \det U^{\ell-1}(u_-(\lambda_\ell)) \det C^{\ell}(u_-(\lambda_\ell), f) \\ = \det C^{\ell-1} \det U^{\ell}(f), \tag{4.11}$$

and (2.4) prove (4.5). The rest then follows from these two equations and Theorem 3.2. □

Remark 4.2. *(i). The ordering of the pairs* (λ_j, γ_j), $1 \le j \le N$ *is clearly irrelevant (interchanging row* i, j *and column* i, j *leaves all determinants unchanged). Moreover, if* $\lambda_i = \lambda_j$, *then* (λ_i, γ_i), (λ_j, γ_j) *can be replaced by* $(\lambda_i, \gamma_i + \gamma_j)$ *(by the first assertion it suffices to verify this for* $N = 2$*).*
(ii). Equation (4.5) can be rephrased as

$$(u_{\gamma_1,\dots,\gamma_N,-}(\lambda_1, x), \dots, u_{\gamma_1,\dots,\gamma_N,-}(\lambda_N, x)) \\ = (C^N(x))^{-1}(u_-(\lambda_1, x), \dots, u_-(\lambda_N, x)), \tag{4.12}$$

where $(C^N(x))^{-1}$ *denotes the inverse matrix of* $C^N(x)$.

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